A GENERALIZATION OF THE BRAUER ALGEBRA

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Abstract. We study two variations of the Brauer algebra $\mathbb{B}_n(x)$. The first is the algebra $\mathbb{A}_n(x)$, which generalizes the Brauer algebra by considering loops. The second is the algebra $\mathbb{L}_n(x)$, the $\mathbb{A}_n(x)$-subalgebra generated by diagrams without horizontal arcs. $\mathbb{A}_n(x)$ and $\mathbb{L}_n(x)$ exhibit for $x \neq 0$ an hereditary-chain indexed by all integers. Following the ideas of Martin [10] in the context of the partition algebra, and Doran *et al.* [4] for the Brauer algebra, we study semisimplicity of $\mathbb{A}_n(x)$ using restriction and induction in $\mathbb{A}_n(x)$ and $\mathbb{L}_n(x)$. Our main result is that $\mathbb{A}_n(x)$ is semisimple if $x \notin \mathbb{Z}$ and that $\mathbb{L}_n(x)$ is semisimple if $x \neq 0$.

1. Introduction

In this paper, we study the semisimplicity of the two diagram algebras $\mathbb{A}_n(x)$ and $\mathbb{L}_n(x)$. $\mathbb{A}_n(x)$ generalizes the Brauer algebra, $\mathbb{B}_n(x)$, by containing diagrams in which vertices can be incident to loops (or equivalently, isolated vertices). $\mathbb{L}_n(x)$ is the $\mathbb{A}_n(x)$-subalgebra generated by all diagrams without any horizontal arcs. The motivation for considering these algebras is twofold: on the one hand in the context of Schur-Weyl duality: $\mathbb{A}_n(x)$ is the centralizer algebra of the group of stochastic, orthogonal matrices and $\mathbb{L}_n(x)$ is the centralizer algebra of the group of stochastic, invertible matrices. On the other hand, $\mathbb{A}_n(x)$ is as the algebra of partial matchings of importance for RNA pseudoknot structures, i.e. helical configurations of RNA primary sequences with cross-serial nucleotide interactions [8].

The Brauer (centralizer) algebras [3] over the field $F = K(x)$, denoted by $\mathbb{B}_n(x)$, are finite dimensional F-algebras indexed by a positive integer n and x, which is either algebraic or transcendent over K. $\mathbb{B}_n(x)$ is the centralizer algebra for the orthogonal or symplectic group on the nth tensor powers of the natural representation. $\mathbb{B}_n(x)$ has been studied by various authors mainly using

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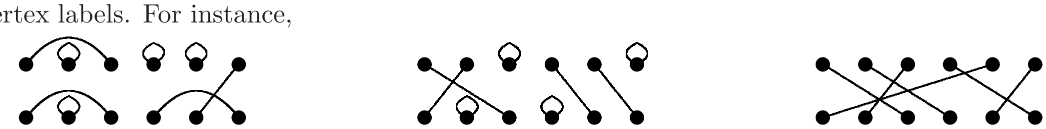
combinatorial methods, see [1, 2, 13, 6, 5, 7] and [15]. Hanlon and Wales conjectured that $\mathbb{B}_n(x)$ is semisimple for all $x \notin \mathbb{Z}$ [6]. Their conjecture was proved by Wenzl [16] and Rui [11] gave necessary and sufficient conditions for the Brauer algebras to be semisimple.

The analysis presented here is based on the concepts of Martin [10] developed in the context of the partition algebra, $\mathbb{P}_n$. Martin's key idea was to relate the existence of certain embeddings to semisimplicity. Subsequently, Doran *et al.* [4] used this framework in order to offer an alternative to Wenzel's proof of semisimplicity. Wenzl's inductive construction hinges on an interpretation of a key ideal in $\mathbb{B}_n(x)$ as the tensor product $\mathbb{B}_{n-1}(x) \otimes_{\mathbb{B}_{n-2}(x)} \mathbb{B}_n(x)$ [9] and the nondegeneracy of a Markov-trace arising naturally in the construction of the latter. The nondegeneracy of this trace form is a result of Weyl's character formulas and is in this sense somewhat "unsatisfactory". The work of Martin [10] and Doran *et al.* [4] puts semisimplicity in the context of quasi-hereditarity and allows to avoid the use of Markov-traces.

Let $\mathscr{A}_n$ be the set of partial 1-factors over $2n$ vertices, i.e. graphs over $2n$ vertices in which each vertex has either degree one zero. We refer to $\mathscr{A}_n$-elements as diagrams and represent them by arranging the $2n$ vertices in two rows, each containing n vertices, with the rows arranged one on top of the other. Furthermore, we equip each isolated vertex with a loop. The n top-vertices are labeled by $[n] = \{1, \ldots, n\}$ in increasing order and the n bottom-vertices are labeled by $[n'] = \{1', \ldots, n'\}$. Let $\mathscr{L}_n \subset \mathscr{A}_n$ be the subset of all $\mathscr{A}_n$-diagrams without any horizontal arcs. We denote the subset of $\mathscr{L}_n$-diagrams having only vertical arcs by $\mathscr{S}_n$. When drawing diagrams, we oftentimes omit vertex labels. For instance,

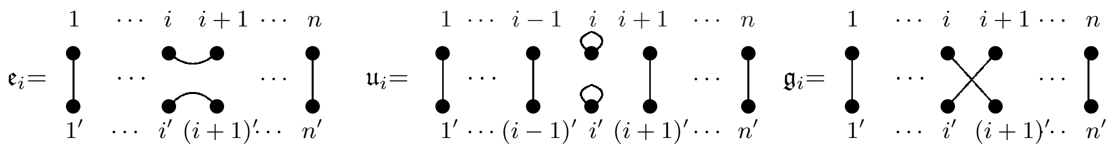


are particular $\mathscr{A}_n$-, $\mathscr{L}_n$- and $\mathscr{S}_n$-diagrams. By abuse of notation, we write S_n instead of $\mathscr{S}_n$, identifying S_n with its embedding into $\mathscr{A}_n$. As for their cardinalities we immediately compute

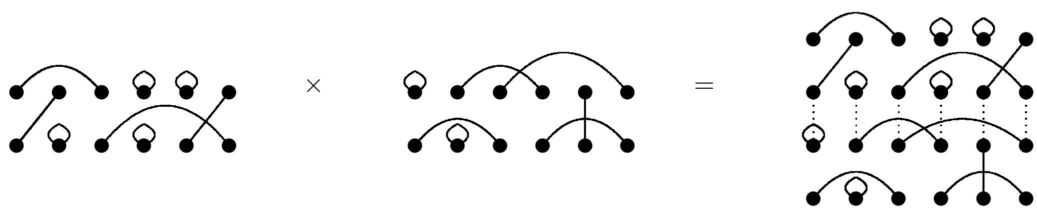
$$|\mathscr{A}_n| = \sum_{j=0}^{n} \binom{2n}{2j} \prod_{i=0}^{(n-j)-1} (2(n-j) - 1 - 2i) \quad \text{and} \quad |\mathscr{L}_n| = \sum_{j=0}^{n} \binom{n}{j}\binom{n}{j} (n-j)!, \tag{1.1}$$

where the factor $\prod_{i=0}^{(n-j)-1}(2(n-j)-1-2i)$ equals the dimension of the Brauer-algebra $\mathbb{B}_{n-j}(x)$. Arcs joining two different vertices, contained both in the top or bottom row are called horizontal arcs. Arcs joining top- and bottom-vertices are called vertical arcs. The induced subgraph of the top and bottom row of a diagram $\mathfrak{a}$ is denoted by $\mathrm{top}(\mathfrak{a})$ and $\mathrm{bot}(\mathfrak{a})$. Let $\mathfrak{e}_i$ be the diagram having

straight verticals except of the horizontal arcs connecting $i, i+1$ and $i', (i+1)'$, $\mathfrak{u}_i$ having straight verticals and loops at i and i' and $\mathfrak{g}_i$ having straight verticals except of the vertical arcs $(i, (i+1)')$ and $(i+1, i')$. Pictorially,



We now describe the multiplication of two diagrams. Let x be either a K-transcendent or algebraic element. We consider $F[\mathscr{A}_n]$, the free F-module generated by $\mathscr{A}_n$ and show that $F[\mathscr{A}_n]$ is a monoid whose multiplication extends that of $\mathbb{B}_n(x)$ in a natural way. To this end let $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}_n$. Let $G(\mathfrak{a}, \mathfrak{b})$ be the graph obtained by arranging the diagram $\mathfrak{a}$ above $\mathfrak{b}$ and introducing the verticals arcs (i, i'), $1 \leq i \leq n$ where i and i' are contained in top($\mathfrak{a}$) and bot($\mathfrak{b}$)-vertex, respectively. For instance,



$G(\mathfrak{a}, \mathfrak{b})$ contains two types of information: (i) $\ell(\mathfrak{a}, \mathfrak{b})$, the number of $G(\mathfrak{a}, \mathfrak{b})$ components that do not contain any vertices of top($\mathfrak{a}$) or bot($\mathfrak{b}$) and (ii) $G'(\mathfrak{a}, \mathfrak{b})$, the graph over the top($\mathfrak{a}$) and bot($\mathfrak{b}$)-vertices obtained as follows: any two vertices are connected by an arc if and only if they are connected by a $G(\mathfrak{a}, \mathfrak{b})$-path. Accordingly, we have $\mathfrak{a} \cdot \mathfrak{b} = x^{\ell(\mathfrak{a}, \mathfrak{b})} G'(\mathfrak{a}, \mathfrak{b})$ and we shall write $\mathfrak{a}\mathfrak{b}$ instead of $\mathfrak{a} \cdot \mathfrak{b}$. $F[\mathscr{A}_n]$ becomes via "$\cdot$" an associative, unitary F-subalgebra of the partition algebra, which we denote by $\mathbb{A}_n(x)$. Furthermore, via "$\cdot$", $F[\mathscr{L}_n]$ becomes an associative F-subalgebra of $\mathbb{A}_n(x)$, denoted by $\mathbb{L}_n(x)$.

By abuse of notation, we write $\mathbb{A}_n = \mathbb{A}_n(x)$, $\mathbb{B}_n = \mathbb{B}_n(x)$ and $\mathbb{L}_n = \mathbb{L}_n(x)$. Furthermore, we shall assume that F is a field of characteristic zero and the term "semisimple" is synonymous to "direct sum of full matrix algebras". In other words, F is a splitting field of $\mathbb{A}_n$ and $\mathbb{L}_n$.

Remark 1. Let $\ell_1(\mathfrak{a}, \mathfrak{b})$ and $\ell_2(\mathfrak{a}, \mathfrak{b})$ denote the number of inner components that are cycles and lines with loops at the start and endpoint. Setting

$$\mathfrak{a} \circ \mathfrak{b} = x_1^{\ell_1(\mathfrak{a}, \mathfrak{b})} \, x_2^{\ell_2(\mathfrak{a}, \mathfrak{b})} \, G'(\mathfrak{a}, \mathfrak{b}), \tag{1.2}$$

we observe that $F[\mathscr{A}_n]$ becomes via "$\circ$" an associative unitary F-algebra, which we denote by $\mathbb{A}_n(x_1, x_2)$. Obviously, in case of $x_1 = x_2$ the multiplications "$\circ$" and "$\cdot$" coincide.

As it is the case for $\mathbb{B}_n$, there exist natural embedding between $\mathbb{A}_{n-1}$ and $\mathbb{A}_n$ obtained by adding the vertices n and n' together with the straight vertical arc, (n, n'), $\epsilon_n \colon \mathbb{A}_{n-1} \longrightarrow \mathbb{A}_n$. By restriction the latter induces an embedding of $\mathbb{L}_{n-1}$ into $\mathbb{L}_n$, which we denote again by $\epsilon_n \colon \mathbb{L}_{n-1} \longrightarrow \mathbb{L}_n$. Furthermore, there exists an involution on $\mathbb{A}_n$ and $\mathbb{L}_n$ obtained by transposing the rows, denoted by $\mathfrak{a} \mapsto \mathfrak{a}^*$. We set $\mathscr{A}_n^m \subset \mathscr{A}_n^n = \mathscr{A}_n$ to be the subset of diagrams having at most m vertical arcs and let $\mathbb{A}_n^m$ be the ideal generated by $\mathscr{A}_n^m$. The ideals $\mathbb{A}_n^m$ for $0 \le m \le n$ give a filtration of $\mathbb{A}_n$, i.e. we have

$$\mathbb{A}_n^0 \subsetneq \mathbb{A}_n^1 \subsetneq \cdots \subsetneq \mathbb{A}_n^{n-1} \subsetneq \mathbb{A}_n^n = \mathbb{A}_n. \tag{1.3}$$

Furthermore, let $\mathbb{I}_n^m = \mathbb{A}_n^m / \mathbb{A}_n^{m-1}$ denote the algebra induced by $\mathbb{A}_n$, which is generated by the set all $\mathscr{A}_n$-diagrams with exactly m vertical arcs, denoted by $\mathscr{I}_n^m$. That is, we have $[\mathfrak{a}] \cdot [\mathfrak{b}] = [\mathfrak{a} \cdot \mathfrak{b}]$ where $[\mathfrak{a} \cdot \mathfrak{b}]$ is zero if it contains less than m vertical arcs. Similarly, we have $[\mathfrak{a}] \circ [\mathfrak{b}] = [\mathfrak{a} \circ \mathfrak{b}]$ in case of "$\circ$". By abuse of notation we shall identify $[\mathfrak{a}]$ with $\mathfrak{a}$. Note that $\mathbb{I}_n^n$ is isomorphic to the group algebra $K[S_n]$. Similarly, $\mathbb{L}_n$ has the filtration

$$\mathbb{L}_n^0 \subsetneq \mathbb{L}_n^1 \subsetneq \cdots \subsetneq \mathbb{L}_n^{n-1} \subsetneq \mathbb{L}_n^n = \mathbb{L}_n \tag{1.4}$$

and by abuse of notation we denote the quotients $\mathbb{L}_n^m / \mathbb{L}_n^{m-1}$ and the set all $\mathscr{L}_n$-diagrams with exactly m vertical arcs again by $\mathbb{I}_n^m$ and $\mathscr{I}_n^m$, respectively.

An integer partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$, where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ is a weakly decreasing sequence of positive integers. If $\sum_i \lambda_i = n$, we write $\lambda \vdash n$. Since any irreducible S_n-module is indexed by a partition [12] λ we write them as S^λ. The dimension of S^λ is denoted by f^λ and its character by χ^λ. The integers λ_i is called the parts of λ. The Ferrer diagram associated with a partition λ is a collection of boxes, $[\lambda]$, in $\mathbb{Z}^2$ using matrix-style coordinates. The boxes are arranged in left-justified rows with weakly decreasing numbers of boxes in each row. For a box $p = (i, j)$ in $[\lambda]$, $j - i$ is the content of p, denoted by $c(p)$. If λ and μ are two partitions such that $\lambda_i \ge \mu_i$ for all i, then we say λ contains μ and write $\mu \subseteq \lambda$. If $\mu \subseteq \lambda$, then the skew partition λ/μ is the set $[\lambda]/[\mu]$. A special case is when λ/μ contains one box only, denoted by $\lambda \sqsupset \mu$. If we identify λ with a Ferrer diagram, then an inner corner of λ is a node $(i, j) \in \lambda$ whose removal leaves the Ferrers diagram of a partition. Any partition μ_1 obtained by such a removal is denoted by $\mu_1 \sqsubset \lambda$. An outer corner of λ is a node $(i, j) \notin \lambda$ whose addition produces the Ferrer diagram of a partition. Any partition μ_2 obtained by such an addition is denoted by $\lambda \sqsubset \mu_2$. Let $\mathrm{res}_{S_{n-1}}^{S_n} S^\lambda$ and $\mathrm{ind}_{S_n}^{S_{n+1}} S^\lambda$ denote the

restriction and the induced representation of S^λ. Then we have [12]

$$\operatorname{res}^{S_n}_{S_{n-1}} S^\lambda \cong \bigoplus_{\mu_1 \sqsubset \lambda} S^{\mu_1} \quad \text{and} \quad \operatorname{ind}^{S_{n+1}}_{S_n} S^\lambda \cong \bigoplus_{\lambda \sqsubset \mu_2} S^{\mu_2}. \tag{1.5}$$

We proceed by describing the induced representation [14] in a specific way. For any $1 \le j \le n-t$, set $\tau_j = (j, n+1-t)$ and $\tau_{n+1-t} = 1$. Then $\{\tau_r \mid 1 \le j \le n+1-t\}$ is a set of representatives of S_{n+1-t}/S_{n-t} and

$$\operatorname{ind}^{S_{n+1-t}}_{S_{n-t}} S^\lambda \cong K[S_{n+1-t}] \otimes_{K[S_{n-t}]} S^\lambda \cong \bigoplus_{j=1}^{n+1-t} (S^\lambda, j). \tag{1.6}$$

Here, the S_{n+1-t}-action on $\operatorname{ind}^{S_{n+1-t}}_{S_{n-t}} S^\lambda$ is given as follows: for given $\sigma \in S_{n+1-t}$ and $1 \le j \le n+1-t$, let s be the unique index such that $\sigma\tau_j \in \tau_s S_{n-t}$ holds, then

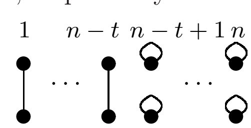
$$\sigma \cdot (w, j) = ((\tau_s^{-1}\sigma\tau_j)w, s). \tag{1.7}$$

In the following, let $\mathbb{X}_n$ denote either $\mathbb{A}_n$ or $\mathbb{L}_n$. Let M be a $\mathbb{X}_n$-left module. Then $\operatorname{res}_{n-1}(M)$ denotes the $\mathbb{X}_{n-1}$-left module, obtained via restriction with respect to the embedding $\epsilon_n \colon \mathbb{X}_{n-1} \longrightarrow \mathbb{X}_n$ and $\operatorname{ind}_{n+1}(M) = \mathbb{X}_{n+1} \otimes_{\mathbb{X}_n} M$ denotes the induced $\mathbb{X}_{n+1}$-left module.

2. $\mathbb{X}_n$-Modules

The semisimplicity of $\mathbb{X}_n$ is closely tied to the structure of $\mathbb{X}_n$-modules. Therefore we shall begin by establishing their basic properties. The latter are a result of the general machinery derived from the fact that $\mathbb{A}_n$ and $\mathbb{L}_n$ are for $x \neq 0$ quasi-hereditary algebras. However, we shall prove them directly. Let $\mathfrak{u}_{n,t}$ denote the diagram having straight verticals except of loops incident to $(n-t+1), \cdots, n$ and $(n-t+1)', \dots, n'$, respectively. Pictorially,

$\mathfrak{u}_{n,t} =$



Let $x \neq 0$ and $\lambda \vdash (n-t) \le n$ be a partition, we set

$$\mathscr{M}_{\mathbb{X}_n}(\lambda) = \mathbb{I}_n^{n-t}\mathfrak{u}_{n,t} \otimes_{S_{n-t}} S^\lambda \quad \text{and} \quad \mathscr{N}_{\mathbb{X}_n}(\lambda) = \{w \in \mathscr{M}_{\mathbb{X}_n}(\lambda) \mid \mathbb{I}_n^{n-t} w = 0\}. \tag{2.1}$$

$\mathscr{M}_{\mathbb{X}_n}(\lambda)$ and $\mathscr{N}_{\mathbb{X}_n}(\lambda)$ become via linear extension of the action

$$\mathfrak{b} \cdot (\mathfrak{a} \otimes v) = (\mathfrak{b}\mathfrak{a}) \otimes v, \tag{2.2}$$

$\mathbb{X}_n$- and $\mathbb{I}_n^{n-t}$-left modules, respectively. Indeed, for any $0 \le t \le n$, $\mathbb{X}_n^{n-t} \lhd \mathbb{X}_n$ is a two sided ideal, which implies that $\mathscr{N}_{\mathbb{X}_n}(\lambda)$ is a $\mathbb{X}_n$-invariant subspace.

Proposition 1. *Let $x \neq 0$ and $\lambda \vdash (n-t) \le n$ be a partition, then the following assertions hold*
(a) *$\mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda)$ is irreducible as a $\mathbb{X}_n$-module and $\mathbb{I}_n^{n-t}$-module, respectively. In particular, $\mathscr{M}_{\mathbb{X}_n}(\lambda)$ is irreducible if and only if $\mathscr{N}_{\mathbb{X}_n}(\lambda) = 0$.*
(b) *$\mathscr{N}_{\mathbb{X}_n}(\lambda)$ is a maximal $\mathbb{X}_n$-submodule of $\mathscr{M}_{\mathbb{X}_n}(\lambda)$ and $\mathscr{N}_{\mathbb{X}_n}(\lambda)$ is unique.*
(c) *For any irreducible $\mathbb{X}_n$-module, V, there exists a partition $\lambda \vdash m \le n$ with the property $V \cong \mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda)$.*

Proof. We first prove (a). Since $\mathscr{N}_{\mathbb{X}_n}(\lambda)$ is a $\mathbb{X}_n$-invariant subspace, $\mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda)$ is a $\mathbb{X}_n$- and $\mathbb{I}_n^{n-t}$-module.
Claim. Any $v \in \mathscr{M}_{\mathbb{X}_n}(\lambda) \setminus \mathscr{N}_{\mathbb{X}_n}(\lambda)$ has the property $\mathbb{I}_n^{n-t} v = \mathscr{M}_{\mathbb{X}_n}(\lambda)$.
To prove the claim we represent $v = \sum_i \mathfrak{a}_i \otimes v_i$, where $\mathfrak{a}_i \in \mathscr{I}_n^{n-t}$ and $\mathrm{bot}(\mathfrak{a}_i) = \mathrm{bot}(\mathfrak{u}_{n,t})$. Let $\delta_{\mathfrak{b}\mathfrak{a}_i} = 1$ if $\mathfrak{b}\mathfrak{a}_i \neq 0$ and $\delta_{\mathfrak{b}\mathfrak{a}_i} = 0$ in $\mathbb{I}_n^{n-t}$, otherwise. For an arbitrary diagram, $\mathfrak{b} \in \mathscr{I}_n^{n-t}$, we have

$$\mathfrak{b} \cdot \sum_i \mathfrak{a}_i \otimes v_i = \sum_i (\mathfrak{b}\mathfrak{a}_i) \otimes v_i = \widetilde{\mathfrak{b}} \otimes \sum_i \delta_{\mathfrak{b}\mathfrak{a}_i} x^{\ell(\mathfrak{a}_i,\mathfrak{b})} \sigma_{\mathfrak{a}_i,\mathfrak{b}} v_i, \tag{2.3}$$

where $\ell(\mathfrak{b}, \mathfrak{a}_i)$ denotes the number of inner components in $G(\mathfrak{b}, \mathfrak{a}_i)$, $\sigma_{\mathfrak{b},\mathfrak{a}_i} \in S_{n-t}$ is such that the diagram $\widetilde{\mathfrak{b}} \in \mathscr{I}_n^{n-t}$ has noncrossing verticals, has $\mathrm{top}(\widetilde{\mathfrak{b}}) = \mathrm{top}(\mathfrak{b})$ and satisfies

$$x^{\ell(\mathfrak{b},\mathfrak{a}_i)} \widetilde{\mathfrak{b}}\, \sigma_{\mathfrak{b},\mathfrak{a}_i} = \mathfrak{b}\mathfrak{a}_i. \tag{2.4}$$

For any $v \in \mathscr{M}_{\mathbb{X}_n}(\lambda) \setminus \mathscr{N}_{\mathbb{X}_n}(\lambda)$ we have $\mathbb{I}_n^{n-t} v \neq 0$, whence there exists some $\mathfrak{b}_0 \in \mathscr{I}_n^{n-t}$ such that

$$\mathfrak{b}_0 \cdot \sum_i \mathfrak{a}_i \otimes v_i = \sum_i (\mathfrak{b}_0\mathfrak{a}_i) \otimes v_i = \widetilde{\mathfrak{b}}_0 \otimes \sum_i \delta_{\mathfrak{b}_0\mathfrak{a}_i} x^{\ell(\mathfrak{b}_0,\mathfrak{a}_i)} \sigma_{\mathfrak{b}_0,\mathfrak{a}_i} v_i \neq 0, \tag{2.5}$$

where $\mathrm{top}(\widetilde{\mathfrak{b}}_0) = \mathrm{top}(\mathfrak{b}_0)$, $\widetilde{\mathfrak{b}}_0 \in \mathscr{I}_n^{n-t}$ has noncrossing verticals and $\sigma_{\mathfrak{b}_0,\mathfrak{a}_i} \in S_{n-t}$ is such that

$$x^{\ell(\mathfrak{b}_0,\mathfrak{a}_i)} \widetilde{\mathfrak{b}}_0\, \sigma_{\mathfrak{b}_0,\mathfrak{a}_i} = \mathfrak{b}_0\mathfrak{a}_i. \tag{2.6}$$

For arbitrary $\mathfrak{b} \in \mathscr{I}_n^{n-t}$ we consider the element $\mathfrak{b}^\ddagger$ having the properties: $\mathrm{top}(\mathfrak{b}^\ddagger) = \mathrm{top}(\mathfrak{b})$, $\mathrm{bot}(\mathfrak{b}^\ddagger) = \mathrm{bot}(\mathfrak{b}_0)$, having $n-t$ vertical arcs and satisfying

$$\mathfrak{b}^\ddagger \mathfrak{a}_i = x^{\ell(\mathfrak{b}_0,\mathfrak{a}_i)} \widetilde{\mathfrak{b}^\ddagger} \sigma_{\mathfrak{b}_0,\mathfrak{a}_i}, \tag{2.7}$$

where $\widetilde{\mathfrak{b}^\ddagger} \in \mathscr{I}_n^{n-t}$ has noncrossing verticals and $\mathrm{top}(\widetilde{\mathfrak{b}^\ddagger}) = \mathrm{top}(\mathfrak{b}^\ddagger) = \mathrm{top}(\mathfrak{b})$. Multiplying with $\mathfrak{b}^\ddagger$ we obtain

$$\mathfrak{b}^\ddagger \cdot \sum_i \mathfrak{a}_i \otimes v_i = \sum_i (\mathfrak{b}^\ddagger\mathfrak{a}_i) \otimes v_i = \widetilde{\mathfrak{b}^\ddagger} \otimes \sum_i \delta_{\mathfrak{b}_0\mathfrak{a}_i} x^{\ell(\mathfrak{b}_0,\mathfrak{a}_i)} \sigma_{\mathfrak{b}_0,\mathfrak{a}_i} v_i \neq 0.$$

We set $w = \sum_i \delta_{\mathfrak{b}_0 \mathfrak{a}_i} x^{\ell(\mathfrak{b}_0, \mathfrak{a}_i)} \sigma_{\mathfrak{b}_0, \mathfrak{a}_i} v_i$ and note that $w \neq 0$ holds. Since S^λ is irreducible, for any $0 \neq u$ the elements $\sigma_0 u$, $\sigma_0 \in S_{n-t}$ generate S^λ. Since for any $\sigma_0 \in S_{n-t}$ there exists some $g(\sigma_0, \mathfrak{b}^\ddagger) \in \mathbb{I}_n^{n-t}$ with the property

$$g(\sigma_0, \mathfrak{b}^\ddagger) \cdot \widetilde{\mathfrak{b}^\ddagger} = x^m \widetilde{\mathfrak{b}^\ddagger} \sigma_0 \quad \text{for some } m \in \mathbb{Z}, \tag{2.8}$$

we conclude

$$g(\sigma_0) \cdot \mathfrak{b}^\ddagger \cdot \sum_i \mathfrak{a}_i \otimes v_i = g(\sigma_0) \cdot \widetilde{\mathfrak{b}^\ddagger} \otimes w = x^m \widetilde{\mathfrak{b}^\ddagger} \otimes \sigma_0 w. \tag{2.9}$$

Accordingly, $\mathbb{I}_n^{n-t} \cdot v = \mathbb{I}_n^{n-t} \mathfrak{u}_{n,t} \otimes_{S_{n-t}} S^\lambda$ and the Claim is proved.

As a result, any nontrivial $\mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda)$-element generates $\mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda)$, which is equivalent to $\mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda)$ being an irreducible $\mathbb{I}_n^{n-t}$-left module. This action extends to an unique $\mathbb{X}_n$-left action with respect to which $\mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda)$ is an irreducible $\mathbb{X}_n$-module. This proves assertion (a).

We next prove (b): the maximality of $\mathscr{N}_{\mathbb{X}_n}(\lambda)$ follows from the irreducibility of $\mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda)$. It remains to show that $\mathscr{N}_{\mathbb{X}_n}(\lambda)$ is unique. For this purpose, let M be a maximal $\mathbb{X}_n$-left submodule of $\mathscr{M}_{\mathbb{X}_n}(\lambda)$ different from $\mathscr{N}_{\mathbb{X}_n}(\lambda)$. Then there exist a $v \in M \setminus \mathscr{N}_{\mathbb{X}_n}(\lambda)$, which, according to (a) generates $\mathscr{M}_{\mathbb{X}_n}(\lambda)$. Consequently, any maximal $\mathscr{M}_{\mathbb{X}_n}(\lambda)$-module, different from $\mathscr{N}_{\mathbb{X}_n}(\lambda)$, is equal to $\mathscr{M}_{\mathbb{X}_n}(\lambda)$ and (b) follows.

Next we show (c). Let $(n-t)$ be the smallest integer with the property $\mathbb{X}_n^{n-t}$ is not acting trivially on V. Consider the set $V_0 = \{v \in V \mid \mathbb{I}_n^{n-t} v = 0\}$. Clearly, since $\mathbb{X}_n^{n-t} \lhd \mathbb{X}_n$ is a two sided ideal, V_0 is an $\mathbb{X}_n$-invariant subspace and the irreducibility of V implies either $V_0 = 0$ or $V_0 = V$. By definition of $(n-t)$, there exists a $v \in V$ such that $\mathbb{I}_n^{n-t} v \neq 0$, whence $V_0 = 0$. Therefore, any $0 \neq v \in V$ has the property $\mathbb{I}_n^{n-t} v \neq 0$ and $\mathbb{I}_n^{n-t} v$ is $\mathbb{A}_n$-invariant. Since V is an irreducible $\mathbb{X}_n$-module we have $\mathbb{I}_n^{n-t} v = V$. Accordingly, V is also an irreducible $\mathbb{I}_n^{n-t}$-left module.

As an $\mathbb{I}_n^{n-t}$-left module the algebra $\mathbb{I}_n^{n-t}$ decomposes into a direct sum of modules that are isomorphic to $\mathscr{M}_{\mathbb{X}_n}(\lambda)$, for $\lambda \vdash (n-t)$, i.e.

$$\mathbb{I}_n^{n-t} \cong \bigoplus_{\lambda \vdash (n-t)} n_\lambda \, \mathscr{M}_{\mathbb{X}_n}(\lambda), \tag{2.10}$$

where n_λ denotes the multiplicity of $\mathscr{M}_{\mathbb{X}_n}(\lambda)$ in $\mathbb{I}_n^{n-t}$. Clearly we have for any $0 \neq v \in V$ the surjective morphism of $\mathbb{I}_n^{n-t}$-left modules $\phi_v \colon \mathbb{I}_n^{n-t} \longrightarrow V$, given by $\mathfrak{a} \mapsto \mathfrak{a} \cdot v$. Accordingly there exists a partition $\lambda \vdash (n-t)$ and a surjective morphism of $\mathbb{I}_n^{n-t}$-left modules induced by ϕ_v:

$$\phi_v^\lambda \colon \mathscr{M}_{\mathbb{X}_n}(\lambda) \longrightarrow V.$$

Assertion (a) and (b) imply $\ker(\phi_v^\lambda) = \mathscr{N}_{\mathbb{X}_n}(\lambda)$, i.e. we have $\mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda) \cong V$ and the proof of Proposition 1 is complete. $\square$

The next result connects semisimplicity of $\mathbb{X}_n$ with the existence of nontrivial morphisms between the modules $\mathscr{M}_{\mathbb{X}_n}(\lambda)$ and $\mathscr{M}_{\mathbb{X}_n}(\mu)$. Indeed, if $\mathbb{X}_n$ is not semisimple, then there exists some module $\mathscr{M}_{\mathbb{X}_n}(\mu)$, $\mu \vdash m < n$ with a nontrivial maximal submodule $\mathscr{N}_{\mathbb{X}_n}(\mu)$. In the following we denote by $\mathrm{Rad}(\mathbb{X}_n)$ the Jacobson radical of $\mathbb{X}_n$, i.e. $\mathbb{X}_n$ is semisimple if and only if $\mathrm{Rad}(\mathbb{X}_n) = 0$.

Proposition 2. *If $\mathbb{X}_n$ is not semisimple, then there exist two partitions μ, λ, where $|\mu| < |\lambda| \leq n$ and a short exact sequence of $\mathbb{X}_n$-modules*

$$0 \longrightarrow \mathscr{N}_{\mathbb{X}_n}(\lambda) \longrightarrow \mathscr{M}_{\mathbb{X}_n}(\lambda) \xrightarrow{\varphi_n} \mathscr{M}_{\mathbb{X}_n}(\mu). \tag{2.11}$$

Proof. Suppose first that $\mathscr{M}_{\mathbb{X}_n}(\mu)$ is for any partition $\mu \vdash m$, $m < n$, irreducible. We claim that $\mathbb{X}_n$ is in this case semisimple. To this end we observe that for $\mu \vdash n$, we have $\mathscr{M}_{\mathbb{X}_n}(\mu) \cong S^\mu$, i.e. for arbitrary partition μ, the module $\mathscr{M}_{\mathbb{X}_n}(\mu)$ is irreducible. In view of

$$\mathbb{I}_n^m \cong \bigoplus_{\mu \vdash m} n_\mu \, \mathscr{M}_{\mathbb{X}_n}(\mu),$$

for any $0 \leq m \leq n$, the F-algebras $\mathbb{X}_n^m / \mathbb{X}_n^{m-1} \cong \mathbb{I}_n^m$ and in particular $\mathbb{X}_n^0 \cong \mathbb{I}_n^0$, are semisimple. Since $\mathrm{Rad}(\mathbb{X}_n^m)$ is a nilpotent ideal so is $(\mathrm{Rad}(\mathbb{X}_n^m) + \mathbb{X}_n^{m-1})/\mathbb{X}_n^{m-1}$ and we obtain

$$(\mathrm{Rad}(\mathbb{X}_n^m) + \mathbb{X}_n^{m-1})/\mathbb{X}_n^{m-1} \subset \mathrm{Rad}(\mathbb{X}_n^m / \mathbb{X}_n^{m-1}) = 0.$$

We next observe using $\mathrm{Rad}(\mathbb{X}_n^m) \cap \mathbb{X}_n^{m-1} = \mathrm{Rad}(\mathbb{X}_n^{m-1})$

$$\begin{aligned} (\mathrm{Rad}(\mathbb{X}_n^m) + \mathbb{X}_n^{m-1})/\mathbb{X}_n^{m-1} &\cong \mathrm{Rad}(\mathbb{X}_n^m)/(\mathrm{Rad}(\mathbb{X}_n^m) \cap \mathbb{X}_n^{m-1}) \\ &\cong \mathrm{Rad}(\mathbb{X}_n^m)/\mathrm{Rad}(\mathbb{X}_n^{m-1}). \end{aligned}$$

Consequently we have for $1 \leq m \leq n$ the inclusion $\mathrm{Rad}(\mathbb{X}_n^m) \subset \mathrm{Rad}(\mathbb{X}_n^{m-1})$, which implies $\mathrm{Rad}(\mathbb{X}_n) \subset \mathrm{Rad}(\mathbb{X}_n^0) = 0$, i.e. $\mathbb{X}_n$ is semisimple.
Thus, if $\mathbb{X}_n$ is not semisimple, there exists a partition $\mu \vdash m$, $m < n$, such that $\mathscr{M}_{\mathbb{X}_n}(\mu)$ is not irreducible. Then there exists according to Proposition 1, assertion (b), the nontrivial, maximal submodule $\mathscr{N}_{\mathbb{X}_n}(\mu) \subset \mathscr{M}_{\mathbb{X}_n}(\mu)$. Let m_0 be the smallest integer such that $\mathbb{X}_n^{m_0}$ acts nontrivially on $\mathscr{N}_{\mathbb{X}_n}(\mu)$. By definition we have for any $v \in \mathscr{N}_{\mathbb{X}_n}(\mu)$, $\mathbb{I}_n^m v = 0$, whence $m < m_0$. $\mathscr{N}_{\mathbb{X}_n}(\mu)$ is then a nontrivial $\mathbb{I}_n^{m_0}$-left module and there exists an irreducible $\mathbb{I}_n^{m_0}$-submodule $W \subset \mathscr{N}_{\mathbb{X}_n}(\mu)$. According to Proposition 1, assertion (c), W is isomorphic to $\mathscr{M}_{\mathbb{X}_n}(\lambda)/\mathscr{N}_{\mathbb{X}_n}(\lambda)$ for some $\lambda \vdash m_0$, i.e. $|\mu| < |\lambda| \leq n$. Therefore there exists a partition λ and a nontrivial morphism of $\mathbb{X}_n$-modules $\varphi_n \colon \mathscr{M}_{\mathbb{X}_n}(\lambda) \longrightarrow \mathscr{M}_{\mathbb{X}_n}(\mu)$, such that $\ker(\varphi_n) = \mathscr{N}_{\mathbb{X}_n}(\lambda)$ and $|\mu| < |\lambda| \leq n$ and the proposition follows. $\square$

3. Restriction and induction

We shall begin by showing that $\mathbb{A}_n$ has the generators $\mathfrak{g}_{i-1}, \mathfrak{e}_{i-1}, \mathfrak{u}_j$, $2 \le i \le n$, $1 \le j \le n$.

Lemma 1. *Any diagram $\mathfrak{a} \in \mathscr{A}_n$ is either contained in $\mathscr{A}_{n-1}$ or of the form*

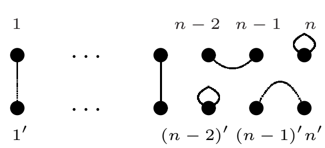
$$\mathfrak{a} = \mathfrak{a}' \, \mathfrak{x} \, \mathfrak{b}', \quad \mathfrak{a}', \mathfrak{b}' \in \mathscr{A}_{n-1}, \ \mathfrak{x} \in \{\mathfrak{g}_{n-1}, \mathfrak{e}_{n-1}, \mathfrak{u}_n\} \ . \tag{3.1}$$

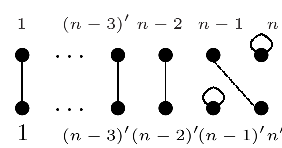
In particular, we have $\mathbb{A}_n = \langle S_n, \mathfrak{e}_{n-1}, \mathfrak{u}_n \rangle$ and $\mathbb{L}_n = \langle S_n, \mathfrak{u}_n \rangle$.

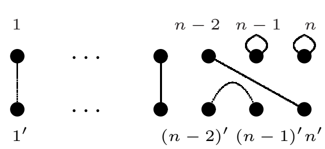
Proof. Any diagram not contained in $\mathscr{A}_{n-1}$ has either (a) none or two loops at the vertices n, n', (b) exactly one loop over $n(n')$ and at least one loop over some vertex $i'(i)$, where $i < n$ or (c) exactly one loop over $n(n')$ and no loops over $i'(i)$, where $i < n$. From this we derive

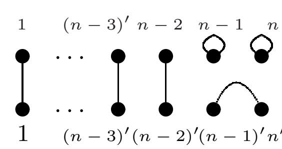
$$\mathfrak{a} = \mathfrak{a}' \, \mathfrak{y} \, \mathfrak{b}' \quad \text{where} \quad \mathfrak{a}', \mathfrak{b}' \in \mathscr{A}_{n-1} \text{ and } \mathfrak{y} \in \begin{cases} \{\mathfrak{g}_{n-1}, \mathfrak{e}_{n-1}, \mathfrak{u}_n\} & \text{(a)} \\ \{\mathfrak{d}_1, \mathfrak{d}_1^*, \mathfrak{d}_2, \mathfrak{d}_2^*\} & \text{(b)} \\ \{\mathfrak{d}_3, \mathfrak{d}_3^*, \mathfrak{d}_4, \mathfrak{d}_4^*\} & \text{(c)} \end{cases} \tag{3.2}$$

where

$\mathfrak{d}_1 =$ 

$\mathfrak{d}_2 =$ 

$\mathfrak{d}_3 =$ 

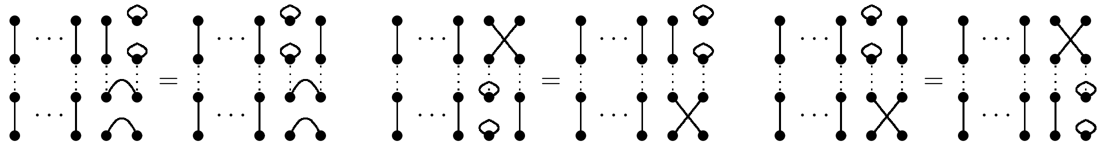
$\mathfrak{d}_4 =$ 

We can express the diagrams $\mathfrak{d}_1, \ldots, \mathfrak{d}_4$ via the generators $\mathfrak{g}_i$, $\mathfrak{e}_i$ and $\mathfrak{u}_i$ as follows

$$\mathfrak{d}_1 = \mathfrak{e}_{n-2} \, \mathfrak{u}_{n-1} \, \mathfrak{u}_{n-2} \, \mathfrak{e}_{n-1}, \quad \mathfrak{d}_2 = \mathfrak{u}_n \, \mathfrak{g}_{n-1}, \quad \mathfrak{d}_3 = \mathfrak{u}_n \, \mathfrak{e}_{n-1} \, \mathfrak{e}_{n-2}, \quad \mathfrak{d}_4 = \mathfrak{u}_{n-1} \, \mathfrak{e}_{n-1}.$$

We next observe that the relations

$$\mathfrak{u}_n \, \mathfrak{e}_{n-1} = \mathfrak{u}_{n-1} \, \mathfrak{e}_{n-1} \qquad \mathfrak{g}_{n-1} \, \mathfrak{u}_{n-1} = \mathfrak{u}_n \, \mathfrak{g}_{n-1} \qquad \mathfrak{u}_{n-1} \, \mathfrak{g}_{n-1} = \mathfrak{g}_{n-1} \, \mathfrak{u}_n$$



imply eq. (3.2) from which the lemma follows. $\square$

The next theorem analyzes the restriction in $\mathbb{A}_n$ and follows the ideas of Doran *et al.* [4] in the case of $\mathbb{B}_n$. We find the following new phenomenon for $\mathbb{A}_n$: for $\lambda \vdash (n-t)$, where $t \geq 1$, there exists an embedding of $\mathscr{M}_{\mathbb{A}_{n-1}}(\lambda)$ into $\mathrm{res}_{n-1}(\mathscr{M}_{\mathbb{A}_n}(\lambda))$. Such an embedding does not exist for $\mathbb{B}_n$. We shall employ it in Lemma 2 in order to show that if $\hom_{\mathbb{X}_n}(\mathscr{M}_{\mathbb{X}_n}(\lambda), \mathscr{M}_{\mathbb{X}_n}(\mu)) \neq 0$ then we can, without loss of generality, assume that $\lambda \vdash n$.

Theorem 1. *Let $n, t \in \mathbb{N}$ and $\lambda \vdash (n-t)$ where $1 \leq t \leq n$. Then there exists the exact sequence of $\mathbb{A}_{n-1}$-modules*

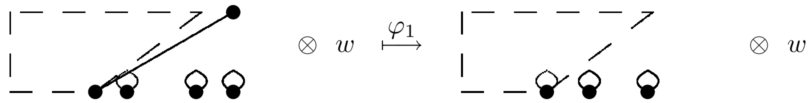
$$0 \longrightarrow \bigoplus_{\alpha \sqsubseteq \lambda} \mathscr{M}_{\mathbb{A}_{n-1}}(\alpha) \longrightarrow \mathrm{res}_{n-1}(\mathscr{M}_{\mathbb{A}_n}(\lambda)) \longrightarrow \bigoplus_{\lambda \sqsubset \beta} \mathscr{M}_{\mathbb{A}_{n-1}}(\beta) \longrightarrow 0. \tag{3.3}$$

Proof. Claim 1. There exists the following short exact sequence of $\mathbb{A}_{n-1}$-left modules

$$0 \longrightarrow \bigoplus_{\alpha \sqsubseteq \lambda} \mathscr{M}_{\mathbb{A}_{n-1}}(\alpha) \longrightarrow \mathrm{res}_{n-1}(\mathscr{M}_{\mathbb{A}_n}(\lambda)). \tag{3.4}$$

Let $F_n^1(\lambda)$ denote the $\mathscr{M}_{\mathbb{A}_n}(\lambda)$-subspace generated by all tensors $\mathfrak{a} \otimes w$, where $\mathfrak{a}$ is a $\mathbb{I}_n^{n-t}\mathfrak{u}_{n,t}$-diagram in which all vertical edges are noncrossing and the top-vertex n is incident to a vertical edge. Obviously, any tensor $\mathfrak{b} \otimes w \in \mathbb{I}_n^{n-t}\mathfrak{u}_{n,t} \otimes_{S_{n-t}} S^\lambda$ in which n is incident to a vertical edge, satisfies $\mathfrak{b} \otimes w = \mathfrak{a} \otimes \sigma w$ for some $\sigma \in S_{n-t}$. Let $f_1(\mathfrak{a})$ be the diagram derived from $\mathfrak{a}$ by removing n and $(n-t)'$ and by shifting all bottom vertices $\ell' > (n-t)'$ down by one. f_1 induces the mapping

$$\begin{aligned} \varphi_1 \colon F_n^1(\lambda) &\longrightarrow \mathbb{I}_{n-1}^{n-1-t}\mathfrak{u}_{n-1,t} \otimes_{S_{(n-1)-t}} \mathrm{res}_{S_{n-1-t}}(S^\lambda) \\ \mathfrak{a} \otimes w &\longmapsto f_1(\mathfrak{a}) \otimes w. \end{aligned} \tag{3.5}$$



We next prove that φ_1 is bijective. Indeed, for any $\mathbb{I}_{n-1}^{n-1-t}\mathfrak{u}_{n-1,t}$-diagram, $\mathfrak{x}$, there exists a unique permutation $\sigma_0 \in S_{n-1-t}$ such that the vertical edges in $\mathfrak{x}\sigma_0$ are noncrossing. Furthermore we have $\mathfrak{x} \otimes w = \mathfrak{x}\sigma_0 \otimes \sigma_0^{-1}w$. Clearly, the tensor $\mathfrak{x}\sigma_0 \otimes \sigma_0^{-1}w$ has a unique φ_1-preimage, $f_1^{-1}(\mathfrak{x}\sigma_0) \otimes \sigma_0^{-1}w$ where $f_1^{-1}(\mathfrak{x}\sigma_0)$ is obtained by shifting the bottom vertices $\ell' \geq (n-t)'$ up by one and by adding the vertices n and $(n-t)'$ together with an vertical edge connecting them. This proves that φ_1 is bijective.

We next show that $F_n^1(\lambda)$ is, via the natural embedding $\epsilon_n \colon \mathbb{A}_{n-1} \longrightarrow \mathbb{A}_n$, an $\mathbb{A}_{n-1}$-module. In view of Lemma 1 it suffices to show

$$\mathfrak{x} \cdot (\mathfrak{a} \otimes v_i) \in F_n^1(\lambda),$$

where $\mathfrak{x} \in \{\sigma, \mathfrak{e}_i, \mathfrak{u}_j\}$, $1 \leq j \leq n-1$, $1 \leq i \leq n-2$ and $\sigma \in S_{n-1}$. Let $\mathfrak{a}$ be a $\mathbb{I}_n^{n-t}\mathfrak{u}_{n,t}$-diagram in which all vertical edges are noncrossing and the top-vertex n is incident to a vertical edge and let $\sigma \in S_{n-1}$. Then there exist a unique $\mathbb{I}_n^{n-t}\mathfrak{u}_{n,t}$-diagram, $\mathfrak{a}'$, with noncrossing vertical edges, in which n is connected to $(n-t)'$ and a permutation $\sigma_0 \in S_{(n-1)-t}$ such that $\sigma\mathfrak{a} = \mathfrak{a}'\sigma_0$ holds. Consequently,

$$\sigma \cdot (\mathfrak{a} \otimes w) = \mathfrak{a}'\sigma_0 \otimes w = \mathfrak{a}' \otimes \sigma_0 w,$$

i.e. $\sigma \cdot (\mathfrak{a} \otimes w) \in F_n^1(\lambda)$. The cases $\mathfrak{e}_i \cdot \mathfrak{a} \otimes v_j$ and $\mathfrak{u}_{i+1} \cdot \mathfrak{a} \otimes v_j$ follow analogously. We next show that φ_1 is an isomorphism of $\mathbb{A}_{n-1}$-modules, that is we prove $\mathfrak{b} \cdot \varphi_1(\zeta) = \varphi_1(\mathfrak{b} \cdot \zeta)$. Indeed, for $\mathfrak{x} \in \{\sigma, \mathfrak{e}_i, \mathfrak{u}_j\}$

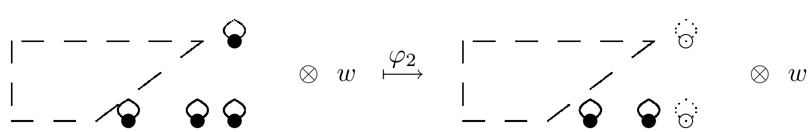
$$\mathfrak{x} \cdot (f(\mathfrak{a}) \otimes w) = f(\mathfrak{x}\,\mathfrak{a}) \otimes w,$$

since neither vertex n or its incident bottom vertex $(n-t)'$ are affected by left multiplication with the elements $\sigma, \mathfrak{e}_i, \mathfrak{u}_j$.

Let $F_n^2(\lambda) \subset \mathscr{M}_{\mathbb{A}_n}(\lambda)$ be the subspace generated by all tensors $\mathfrak{a} \otimes v_i$, where $\mathfrak{a} \in \mathbb{I}_n^{n-t}\mathfrak{u}_{n,t}$ is a diagram having a loop at vertex n. Let $f_2(\mathfrak{a}) \in \mathbb{I}_{n-1}^{n-t}\mathfrak{u}_{n-1,t-1}$ be the diagram obtained by removing the vertices n and n' together with their loops. It is straightforward to show that f_2 induces the isomorphism of $\mathbb{A}_{n-1}$-modules

$$\begin{aligned} \varphi_2 \colon F_n^2(\lambda) &\longrightarrow \mathbb{I}_{n-1}^{n-t}\mathfrak{u}_{n-1,t-1} \otimes_{S_{n-t}} S^\lambda \\ \mathfrak{a} \otimes w &\longmapsto f_2(\mathfrak{a}) \otimes w, \end{aligned} \tag{3.6}$$

where $\mathbb{I}_{n-1}^{n-t}\mathfrak{u}_{n-1,t-1} \otimes_{S_{n-t}} S^\lambda \cong \mathscr{M}_{\mathbb{A}_{n-1}}(\lambda)$.



In view of $\mathrm{res}_{S_{n-1-t}}(S^\lambda) \cong \bigoplus_{\alpha \sqsubset \lambda} S^\alpha$ we derive

$$\begin{aligned} F_n^1(\lambda) \oplus F_n^2(\lambda) &\cong \left[\mathbb{I}_{n-1}^{n-1-t}\mathfrak{u}_{n-1,t} \otimes_{S_{(n-1)-t}} \mathrm{res}_{S_{n-1-t}}(S^\lambda)\right] \oplus \left[\mathbb{I}_{n-1}^{n-t}\mathfrak{u}_{n-1,t-1} \otimes_{S_{n-t}} S^\lambda\right] \\ &\cong \bigoplus_{\alpha \sqsubset \lambda} \left[\mathbb{I}_{n-1}^{n-1-t}\mathfrak{u}_{n-1,t} \otimes_{S_{(n-1)-t}} S^\alpha\right] \oplus \left[\mathbb{I}_{n-1}^{n-t}\mathfrak{u}_{n-1,t-1} \otimes_{S_{n-t}} S^\lambda\right], \end{aligned}$$

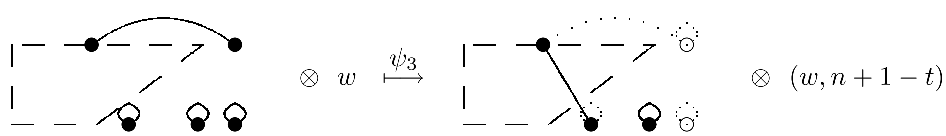
which gives rise to the short exact sequence $0 \longrightarrow \bigoplus_{\alpha \sqsubset \lambda} \mathscr{M}_{\mathbb{A}_{n-1}}(\alpha) \longrightarrow \mathrm{res}_{n-1}(\mathscr{M}_{\mathbb{A}_n}(\lambda))$ and Claim 1 follows.

Claim 2. Let $F_n(\lambda) = F_n^1(\lambda) \oplus F_n^2(\lambda)$, then we have an isomorphism of $\mathbb{A}_{n-1}$-left modules

$$\mathrm{res}_{n-1}\left(\mathscr{M}_{\mathbb{A}_n}(\lambda)/F_n(\lambda)\right) \cong \bigoplus_{\lambda \sqsubset \beta} \mathscr{M}_{\mathbb{A}_{n-1}}(\beta). \tag{3.7}$$

Let $G_n(\lambda)$ denote the space generated by all tensors of the form $\mathfrak{c} \otimes w$, where $\mathfrak{c} \in \mathbb{I}_n^{n-t}\mathfrak{u}_{n,t}$ is a diagram with noncrossing vertical arcs and a horizontal arc incident to n. Let $f_3(\mathfrak{c})$ be the diagram obtained from $\mathfrak{c}$ as follows: one removes n together with its incident horizontal arc and the bottom-vertex n' together with its incident loop. This leaves a unique top-vertex, r, isolated. Next one removes the loop of the bottom-vertex $(n-t+1)'$ and connects it to r via a vertical arc. We next show that f_3 induces the bijection

$$\begin{aligned} \varphi_3\colon \mathrm{res}_{n-1}\left(\mathscr{M}_{\mathbb{A}_n}(\lambda)\right)/F_n(\lambda)) &\longrightarrow \mathbb{I}_{n-1}^{n+1-t}\mathfrak{u}_{n-1,t-2} \otimes_{S_{n+1-t}} \mathrm{ind}_{S_{n-t}}^{S_{n+1-t}}(S^\lambda) \\ \mathfrak{c} \otimes w &\longmapsto f_3(\mathfrak{c}) \otimes (w, n+1-t). \end{aligned} \tag{3.8}$$



Recall that for any $1 \leq j \leq n-t$, $\tau_j = (j, n+1-t)$ and $\tau_{n+1-t} = 1$. Then $S_{n+1-t} = \dot{\bigcup}\tau_j S_{n-t}$, i.e. the τ_r form a set of representatives of S_{n+1-t}/S_{n-t}. We inspect that there exists some $\sigma \in S_{n-t+1}$ such that $f_3(\mathfrak{c})\sigma^{-1} = \tilde{\mathfrak{c}}$ has noncrossing vertical arcs. Then we have $\sigma = \tau_j\sigma_0$, for some $\sigma_0 \in S_{n-t}$. Therefore, in view of $f_3(\mathfrak{c})\sigma^{-1} = \tilde{\mathfrak{c}}$, each $f_3(\mathfrak{c})$ gives rise to some unique τ_j. Using eq. (1.7) we obtain

$$\begin{aligned} f_3(\mathfrak{c})\sigma^{-1}\sigma \otimes (w, n+1-t) &= \tilde{\mathfrak{c}}\tau_j\sigma_0 \otimes (w, n+1-t) \\ &= \tilde{\mathfrak{c}}\tau_j \otimes (\sigma_0 w, n+1-t) \\ &= \tilde{\mathfrak{c}} \otimes (\sigma_0 w, j). \end{aligned}$$

There exist exactly $(n+1-t)$ different $\mathscr{I}_n^{n-t}$-diagrams $\mathfrak{c}_1, \dots, \mathfrak{c}_{n+1-t}$ having noncrossing vertical arcs in which n is connected to a top-vertex and $\mathrm{bot}(\mathfrak{c}_j) = \mathrm{bot}(\mathfrak{u}_{n,t})$ with the property

$$f_3(\mathfrak{c}_j)\sigma^{(j)} = \tilde{\mathfrak{c}} \tag{3.9}$$

for some $\sigma^{(j)} \in S_{n-t+1}$. Since $\dim[\mathrm{ind}_{S_{n-t}}^{S_{n+1-t}}(S^\lambda)] = (n+1-t)\cdot\dim[S^\lambda]$, we obtain

$$\dim\left[\mathrm{res}_{n-1}\left(\mathscr{M}_{\mathbb{A}_n}(\lambda)/F_n(\lambda)\right)\right] = \dim\left[\mathbb{I}_{n-1}^{n+1-t}\mathfrak{u}_{n-1,t-2} \otimes_{S_{n+1-t}} \mathrm{ind}_{S_{n-t}}^{S_{n+1-t}}(S^\lambda)\right]. \tag{3.10}$$

Therefore it suffices to prove that φ_3 is surjective. $\mathbb{I}_{n-1}^{n+1-t}\mathfrak{u}_{n-1,t-2} \otimes_{S_{n+1-t}} \mathrm{ind}_{S_{n-t}}^{S_{n+1-t}}(S^\lambda)$ is generated by tensors of the form $\mathfrak{d}\otimes(w,j)$, where $1 \le j \le n+1-t$, $\mathfrak{d} \in \mathscr{I}_{n-1}^{n-t+1}$ with noncrossing vertical arcs, $\mathrm{bot}(\mathfrak{d}) = \mathrm{bot}(\mathfrak{u}_{n-1,t-2})$ and $w \in S^\lambda$. Since for $1 \le j \le n+1-t$, we have $\tau_j\cdot(w, n+1-t) = (w,j)$ we obtain

$$\mathfrak{d}\otimes(w,j) = \mathfrak{d}\otimes\tau_j\cdot(w,n+1-t) = \mathfrak{d}\tau_j\otimes(w,n+1-t). \tag{3.11}$$

By construction $\mathfrak{d}\tau_j$ is a diagram in which $(n+1-t)'$ connected to a top vertex, which we denote by r. Then there exists some $\sigma_0 \in S_{n-t}$ such that in $\mathfrak{d}\tau_j\sigma_0$ any pair of crossing verticals contains the vertical arc $((n+1-t)', r)$. Let $\mathfrak{c} \in \mathscr{I}_n^{n-t}$, be derived from $\mathfrak{d}\tau_j\sigma_0$ by removing $(r, (n+1-t)')$, adding the vertices n and n', the loops at $(n+1-t)'$ and n', as well as the horizontal arc (r,n). By construction $\mathrm{bot}(\mathfrak{c}) = \mathrm{bot}(\mathfrak{u}_{n,t})$, $\mathfrak{c}$ has noncrossing verticals and we have

$$(\mathfrak{d}\tau_j)\sigma_0 = f_3(\mathfrak{c}). \tag{3.12}$$

Consequently, using the fact that the tensor product is over S_{n+1-t}

$$\begin{aligned}
\mathfrak{d}\otimes(w,j) &= \mathfrak{d}\tau_j\otimes(w,n+1-t)\\
&= f_3(\mathfrak{c})\sigma_0^{-1}\otimes(w,n+1-t)\\
&= f_3(\mathfrak{c})\otimes(\sigma_0^{-1}w,n+1-t),
\end{aligned}$$

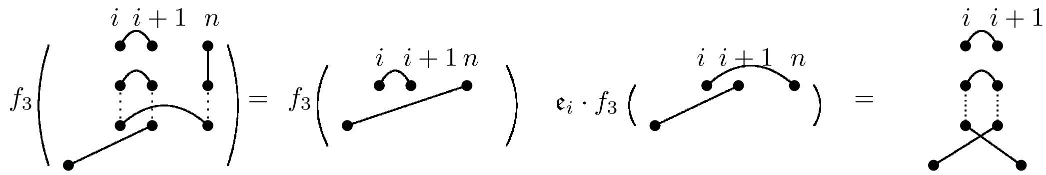
which proves that φ_3 is surjective. We proceed by showing that φ_3 is an isomorphism of $\mathbb{A}_{n-1}$-modules. Since any $\sigma \in S_{n-1}$ fixes n we inspect

$$\forall\, \sigma \in S_{n-1}; \quad \varphi_3(\sigma\cdot\mathfrak{c}\otimes w) = \sigma\cdot f_3(\mathfrak{c})\otimes(w,n+1-t) = \sigma\cdot\varphi_3(\mathfrak{c}\otimes w). \tag{3.13}$$

We next consider the action of $\mathfrak{e}_i$, $1 \le i \le n-2$. Suppose n is connected to r in $\mathfrak{c}$ and $r \ne i+1, i$. Then we immediately obtain

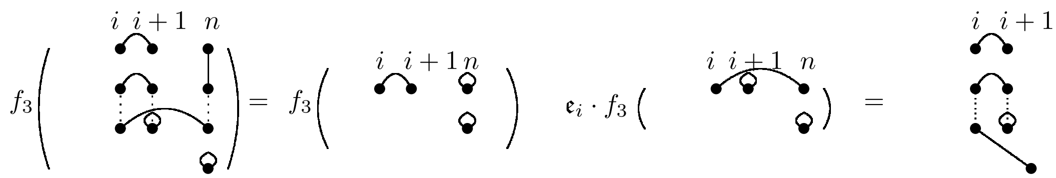
$$\varphi_3(\mathfrak{e}_i\cdot\mathfrak{c}\otimes w) = \mathfrak{e}_i\cdot f_3(\mathfrak{c})\otimes(w,n+1-t) = \mathfrak{e}_i\cdot\varphi_3(\mathfrak{c}\otimes w). \tag{3.14}$$

Without loss of generality we may assume $r = i$. We distinguish three cases:
(1) if $i+1$ is incident to a vertical arc, in $\mathfrak{e}_i\mathfrak{c}$ the top-vertex n is connected to a bottom vertex, whence $\mathfrak{e}_i\cdot\mathfrak{c}\otimes w \equiv 0$ modulo $F_n(\lambda)$,



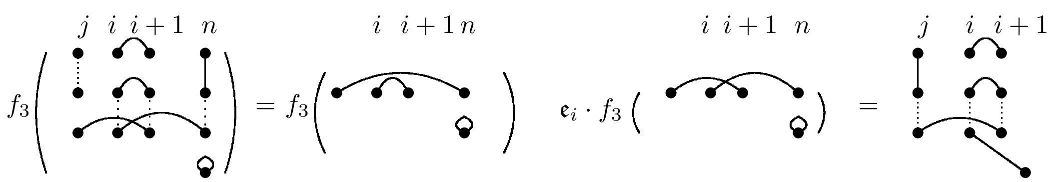
On the other hand, in $f_3(\mathfrak{c})$, $i+1$ and i are connected to vertical arcs, whence $\mathfrak{e}_i \cdot \varphi_3(\mathfrak{c} \otimes w)$ has fewer than $(n+1-t)$ vertical arcs and is consequently zero in $\mathbb{I}_{n-1}^{n+1-t}\mathfrak{u}_{n-1,t-2}$.
(2) if $i+1$ is incident to a loop, n is incident to a loop in $\mathfrak{e}_i\mathfrak{c}$. Clearly we then have $f_3(\mathfrak{e}_i\mathfrak{c}) = \mathfrak{e}_i f_3(\mathfrak{c})$ implying

$$\varphi_3(\mathfrak{e}_i \cdot \mathfrak{c} \otimes w) = \mathfrak{e}_i \cdot f_3(\mathfrak{c}) \otimes (w, n+1-t) = \mathfrak{e}_i \cdot \varphi_3(\mathfrak{c} \otimes w).$$



(3) if $i+1$ is incident to j via a horizontal arc, n is connected to j in $\mathfrak{e}_i\mathfrak{c}$. Clearly we then have $f_3(\mathfrak{e}_i\mathfrak{c}) = \mathfrak{e}_i f_3(\mathfrak{c})$ implying

$$\varphi_3(\mathfrak{e}_i \cdot \mathfrak{c} \otimes w) = \mathfrak{e}_i \cdot f_3(\mathfrak{c}) \otimes (w, n+1-t) = \mathfrak{e}_i \cdot \varphi_3(\mathfrak{c} \otimes w).$$



Finally we consider the action of $\mathfrak{u}_i$, $1 \leq i \leq n-1$. Suppose first $r \neq i$. By definition of f_3, a vertex $i \neq r$ is in $\mathfrak{c}$ incident to a vertical arc if and only if this holds for $f_3(\mathfrak{c})$. In this case we have $\mathfrak{u}_i\mathfrak{c} \equiv 0 \mod F_n(\lambda)$ and $\mathfrak{u}_i f_3(\mathfrak{c}) \equiv 0$ in $\mathbb{I}_{n-1}^{n+1-t}\mathfrak{u}_{n-1,t-2}$. If i is incident to a loop we have $\mathfrak{u}_i\mathfrak{c} = x\mathfrak{c}$ and $\mathfrak{u}_i f_3(\mathfrak{c}) = x f_3(\mathfrak{c})$, i.e. $\varphi_3(\mathfrak{u}_i \cdot \mathfrak{c} \otimes w) = \mathfrak{u}_i \cdot \varphi_3(\mathfrak{c} \otimes w)$. Finally, if i is incident to a horizontal arc we have $f_3(\mathfrak{u}_i\mathfrak{c}) = \mathfrak{u}_i f_3(\mathfrak{c})$. Second let $r = i$. On the one hand we obtain $\mathfrak{u}_i\mathfrak{c} \equiv 0$ modulo $F_n(\lambda)$, since the i'-loop of $\mathfrak{u}_i$ traces back to the top vertex n of $\mathfrak{u}_i\mathfrak{c}$. On the other hand, in $\mathfrak{u}_i f_3(\mathfrak{c})$ the i'-loop of $\mathfrak{u}_i$ traces back to the bottom vertex $(n+1-t)'$. Consequently, $\mathfrak{u}_i f_3(\mathfrak{c})$ has fewer than $(n+1-t)$ vertical arcs and is zero in $\mathbb{I}_{n-1}^{n+1-t}\mathfrak{u}_{n-1,t-2}$. Therefore φ_3 is an isomorphism of $\mathbb{A}_{n-1}$-left

modules. In view of $\mathrm{ind}_{S_{n-t}}^{S_{n+1-t}}(S^\lambda) \cong \bigoplus_{\lambda \sqsubset \beta} S^\beta$ we derive

$$\begin{aligned} \mathrm{res}_{n-1}\left(\mathscr{M}_{\mathbb{A}_n}(\lambda))/F_n(\lambda)\right) &\cong \mathbb{I}_{n-1}^{n+1-t}\mathfrak{u}_{n-1,t-2} \otimes_{S_{n+1-t}} \mathrm{ind}_{S_{n-t}}^{S_{n+1-t}}(S^\lambda) \\ &\cong \bigoplus_{\lambda \sqsubset \beta} \left(\mathbb{I}_{n-1}^{n+1-t}\mathfrak{u}_{n-1,t-2} \otimes_{S_{n+1-t}} S^\beta\right) \\ &\cong \bigoplus_{\lambda \sqsubset \beta} \mathscr{M}_{\mathbb{A}_{n-1}}(\beta) \end{aligned}$$

and the proof of the theorem is complete. $\square$

For $\mathbb{L}_n$, there exists no nontrivial space $G_n(\lambda)$ and Theorem 1 accordingly implies

Corollary 1. *Let $n, t \in \mathbb{N}$ and $\lambda \vdash (n-t)$ where $1 \leq t \leq n$. Then we have the isomorphism of $\mathbb{L}_{n-1}$-modules*

$$\bigoplus_{\alpha \sqsubseteq \lambda} \mathscr{M}_{\mathbb{L}_{n-1}}(\alpha) \cong \mathrm{res}_{n-1}(\mathscr{M}_{\mathbb{L}_n}(\lambda)). \tag{3.15}$$

We proceed by studying induction in $\mathbb{A}_n$. Let us begin by remarking that the arguments of the following proof can easily be put into context with the localization and globalization functors [10, 4]. Since the latter are compatible with the quasi-hereditary structure of $\mathbb{A}_n$, in case of $x \neq 0$ one can obtain a more structural point of view.

Theorem 2. *Let $n, t \in \mathbb{N}$ and $\lambda \vdash (n-t)$ where $1 \leq t \leq n$. Then we have*

$$\mathrm{ind}_{n+1}(\mathscr{M}_{\mathbb{A}_n}(\lambda)) \cong \mathrm{res}_{n+1}(\mathscr{M}_{\mathbb{A}_{n+2}}(\lambda)). \tag{3.16}$$

Furthermore there exists the exact sequence of $\mathbb{A}_{n+1}$-modules

$$0 \longrightarrow \bigoplus_{\alpha \sqsubseteq \lambda} \mathscr{M}_{\mathbb{A}_{n+1}}(\alpha) \longrightarrow \mathrm{ind}_{n+1}(\mathscr{M}_{\mathbb{A}_n}(\lambda)) \longrightarrow \bigoplus_{\lambda \sqsubset \beta} \mathscr{M}_{\mathbb{A}_{n+1}}(\beta) \longrightarrow 0. \tag{3.17}$$

Proof. We first prove eq. (3.16). Suppose we have $\mathfrak{a} \in \mathscr{A}_{n+2}$, with the property that its bottom vertices $(n+1)'$ and $(n+2)'$ are connected by a horizontal arc. Let $f_4(\mathfrak{a})$ be the diagram obtained from $\mathfrak{a}$ by removing its bottom vertices $(n+1)'$, $(n+2)'$ together with their horizontal arc and moving its top-vertex $(n+2)$ to the bottom at position $(n+1)'$. It is straightforward to prove that for any $\mathscr{M}_{\mathbb{A}_n}(\lambda)$ the mapping

$$\begin{aligned} \varphi_4 \colon \mathrm{res}_{n+1}(\mathbb{A}_{n+2}\mathfrak{e}_{n+1} \otimes_{\mathbb{A}_n} \mathscr{M}_{\mathbb{A}_n}(\lambda)) &\longrightarrow \mathbb{A}_{n+1} \otimes_{\mathbb{A}_n} \mathscr{M}_{\mathbb{A}_n}(\lambda) \\ \mathfrak{a} \otimes w &\longmapsto f_4(\mathfrak{a}) \otimes w, \end{aligned} \tag{3.18}$$

is an isomorphism of $\mathbb{A}_{n+1}$-modules. We proceed by showing

$$\mathbb{A}_{n+2}\mathfrak{e}_{n+1} \otimes_{\mathbb{A}_n} \mathscr{M}_{\mathbb{A}_n}(\lambda) \cong \mathscr{M}_{\mathbb{A}_{n+2}}(\lambda). \tag{3.19}$$

The key to eq. (3.19) is to prove that

$$\mathbb{A}_{n+2}\mathfrak{e}_{n+1} \otimes_{\mathbb{A}_n} \mathbb{I}_n^{n-t}\mathfrak{u}_{n,t} \cong \mathbb{I}_{n+2}^{n-t}\mathfrak{u}_{n+2,t+2} \tag{3.20}$$

is an isomorphism of $\mathbb{A}_{n+2}$-left modules. For this purpose we consider a tensor $\mathfrak{a}\mathfrak{e}_{n+1} \otimes \mathfrak{b}\mathfrak{u}_{n,t}$, where $\mathfrak{a}\mathfrak{e}_{n+1} \in \mathbb{A}_{n+2}\mathfrak{e}_{n+1}$ and $\mathfrak{b} \in \mathscr{A}_n^{n-t}$. Let $\mathfrak{x} \in \mathscr{A}_n^{n-t}$ be obtained from $\mathfrak{b}$ as follows: we set $\mathrm{bot}(\mathfrak{x}) = \mathrm{top}(\mathfrak{b})$, $\mathrm{top}(\mathfrak{x}) = \mathrm{top}(\mathfrak{u}_{n,t})$ and choose the vertical $\mathfrak{x}$-arcs and $m \in \mathbb{Z}$ such that

$$x^m\, \mathfrak{x}\, \mathfrak{b}\, \mathfrak{u}_{n,t} = \mathfrak{u}_{n,t}. \tag{3.21}$$

Since the product $\mathfrak{x}^*\mathfrak{x}$ generates exactly t inner components, we obtain using eq. (3.21)

$$x^{-t+m}\, \mathfrak{x}^*\mathfrak{x}\, \mathfrak{b}\, \mathfrak{u}_{n,t} = \mathfrak{b}\, \mathfrak{u}_{n,t}.$$

Using $\mathrm{bot}(\mathfrak{x}^*) = \mathrm{top}(\mathfrak{x}) = \mathrm{top}(\mathfrak{u}_{n,t})$, we compute

$$\begin{aligned}
\mathfrak{a}\mathfrak{e}_{n+1} \otimes \mathfrak{b}\, \mathfrak{u}_{n,t} &= \mathfrak{a}\mathfrak{e}_{n+1} \otimes x^{-t+m}\, \mathfrak{x}^*\mathfrak{x}\, \mathfrak{b}\, \mathfrak{u}_{n,t} \\
&= \mathfrak{a}\mathfrak{e}_{n+1}x^{-t}\, \mathfrak{x}^* \otimes x^m\, \mathfrak{x}\, \mathfrak{b}\, \mathfrak{u}_{n,t} \\
&= \mathfrak{a}\mathfrak{e}_{n+1}x^{-t}\, \mathfrak{x}^* \otimes \mathfrak{u}_{n,t} \\
&= x^{-t}\, \mathfrak{a}\, \mathfrak{x}^*\mathfrak{e}_{n+1} \otimes \mathfrak{u}_{n,t} \\
&= \mathfrak{a}'\mathfrak{u}_{n,t}\mathfrak{e}_{n+1} \otimes \mathfrak{u}_{n,t}.
\end{aligned}$$

Employing the just derived normal form for tensors, we are now in position to make the isomorphism of $\mathbb{A}_{n+2}$-left modules of eq. (3.20) explicit

$$\begin{aligned}
\varphi_5\colon \mathbb{A}_{n+2}\mathfrak{e}_{n+1} \otimes_{\mathbb{A}_n} \mathbb{I}_n^{n-t}\mathfrak{u}_{n,t} &\longrightarrow \mathbb{I}_{n+2}^{n-t}\mathfrak{u}_{n+2,t+2} \\
\mathfrak{a}'\mathfrak{u}_{n,t}\mathfrak{e}_{n+1} \otimes \mathfrak{u}_{n,t} &\longmapsto \mathfrak{a}'\mathfrak{u}_{n+2,t+2}.
\end{aligned}$$

Standard tensor identities imply

$$\begin{aligned}
\mathbb{A}_{n+2}\mathfrak{e}_{n+1} \otimes_{\mathbb{A}_n} \mathscr{M}_{\mathbb{A}_n}(\lambda) &\cong (\mathbb{A}_{n+2}\mathfrak{e}_{n+1} \otimes_{\mathbb{A}_n} \mathbb{I}_n^{n-t}\mathfrak{u}_{n,t}) \otimes_{S_{n-t}} S^\lambda \\
&\cong \mathbb{I}_{n+2}^{n-t}\mathfrak{u}_{n+2,t+2} \otimes_{S_{n-t}} S^\lambda \\
&\cong \mathscr{M}_{\mathbb{A}_{n+2}}(\lambda).
\end{aligned}$$

Now Claim 3 follows immediately

$$\begin{aligned}
\mathrm{ind}_{n+1}(\mathscr{M}_{\mathbb{A}_n}(\lambda)) &= \mathbb{A}_{n+1} \otimes_{\mathbb{A}_n} \mathscr{M}_{\mathbb{A}_n}(\lambda) \\
&\cong \mathrm{res}_{n+1}(\mathbb{A}_{n+2}\mathfrak{e}_{n+1} \otimes_{\mathbb{A}_n} \mathscr{M}_{\mathbb{A}_n}(\lambda)) \\
&\cong \mathrm{res}_{n+1}(\mathscr{M}_{\mathbb{A}_{n+2}}(\lambda)).
\end{aligned}$$

Accordingly, the exact sequence of eq. (3.17) is immediately implied by Theorem 1 and the proof of the theorem is complete. □

Corollary 2. *Let $n, t \in \mathbb{N}$ and $\lambda \vdash (n-t)$ where $1 \leq t \leq n$. Then we have*

$$\mathrm{ind}_{n+1}(\mathscr{M}_{\mathbb{L}_n}(\lambda)) \cong \mathrm{res}_{n+1}(\mathscr{M}_{\mathbb{L}_{n+2}}(\lambda)) \quad \text{and} \quad \bigoplus_{\alpha \sqsubseteq \lambda} \mathscr{M}_{\mathbb{L}_{n+1}}(\alpha) \cong \mathrm{ind}_{n+1}(\mathscr{M}_{\mathbb{L}_n}(\lambda)). \tag{3.22}$$

4. Semisimplicity

The semisimplicity of $\mathbb{L}_n$ is an immediate consequence of Proposition 1 and Proposition 2.

Theorem 3. *Suppose $x \neq 0$, then $\mathbb{L}_n$ is semisimple.*

Proof. We showed in Proposition 2, that if $\mathbb{L}_n$ is not semisimple, then there exist two partitions μ, λ, where $|\mu| < |\lambda| \leq n$ and a nontrivial morphism of $\mathbb{L}_n$-modules $\mathscr{M}_{\mathbb{L}_n}(\lambda) \xrightarrow{\varphi_n} \mathscr{M}_{\mathbb{L}_n}(\mu)$. The uniqueness of $\mathscr{N}_{\mathbb{L}_n}(\mu)$ implies that $\varphi_n(\mathscr{M}_{\mathbb{L}_n}(\lambda)) \subset \mathscr{N}_{\mathbb{L}_n}(\mu)$.
Claim. For $x \neq 0$ we have $\mathscr{N}_{\mathbb{L}_n}(\mu) = 0$.
In case of $\mu \vdash n$ this follows immediately from the irreducibility of the lift of the Specht module S^{λ}. Suppose next $\mu \vdash (n-t) < n$. Let $\mathfrak{a} \in \mathscr{I}_n^{n-t}$, where $\mathrm{bot}(\mathfrak{a}) = \mathrm{bot}(\mathfrak{u}_{n,t})$ and let $v \in S^{\mu}$. For any $\mathfrak{a} \otimes v \in \mathscr{M}_{\mathbb{L}_n}(\mu)$, there exists some $\sigma_0 \in S_{n-t}$ and some t-tuple $(j_1, j_2, \dots, j_t)$, where $1 \leq j_1 < j_2 < \cdots < j_t \leq n$ such that $\mathfrak{a}_{(j_h)_{h=1}^t} = \mathfrak{a}\sigma_0$ has noncrossing vertical arcs and has top-vertex loops at $j_1, \dots, j_t$. The $\mathscr{I}_n^{n-t}$-diagram, $\mathfrak{a}_{(j_h)_{h=1}^t}$ has the property $\mathfrak{a} \otimes v = \mathfrak{a}_{(j_h)_{h=1}^t} \otimes \sigma_0^{-1} v$ and any $u \in \mathbb{I}_n^{n-t}\mathfrak{u}_{n,t} \otimes_{S_{n-1}} S^{\mu}$ can be written as

$$u = \sum_{1 \leq j_1 < j_2 < \cdots < j_t \leq n} \mathfrak{a}_{(j_h)_{h=1}^t} \otimes w_{(j_h)_{h=1}^t}.$$

For $U_t = \sum_{1 \leq i_1 < i_2 < \cdots < i_t \leq n} \mathfrak{u}_{i_1} \cdots \mathfrak{u}_{i_t} \in \mathbb{I}_n^{n-t}$ we immediately obtain $U_t \cdot \mathfrak{a}_{(j_h)_{h=1}^t} = x^t\, \mathfrak{a}_{(j_h)_{h=1}^t}$. Indeed, only if $(i_1, i_2, \dots, i_t)$ matches the tuple $(j_h)_{h=1}^t$ the factor x^t via the t-inner components of the graph $G(\mathfrak{u}_{i_j} \cdots \mathfrak{u}_{i_t}, \mathfrak{a}_{(j_h)_{h=1}^t})$ is produced. In all other cases there exists a loop which traces back to the bottom row of $G'(\mathfrak{u}_{i_j} \cdots \mathfrak{u}_{i_t}, \mathfrak{a}_{(j_h)_{h=1}^t})$ resulting in a zero in $\mathbb{I}_n^{n-t}$. Therefore, for any $u \in \mathscr{M}_{\mathbb{L}_n}(\mu)$

$$U_t \cdot u = x^t\, u \tag{4.1}$$

holds. Since $U_t \in \mathbb{I}_n^{n-t}$, $x \neq 0$ implies $\mathscr{N}_{\mathbb{L}_n}(\mu) = \{w \in \mathscr{M}_{\mathbb{L}_n}(\mu) \mid \mathbb{I}_n^{n-t} w = 0\} = 0$ and the Claim is proved.

The uniqueness of $\mathscr{N}_{\mathbb{L}_n}(\mu)$ as a maximal $\mathscr{M}_{\mathbb{L}_n}(\mu)$ module implies that any nontrivial morphism φ_n has the property $\varphi_n(\mathscr{M}_{\mathbb{L}_n}(\lambda)) \subset \mathscr{N}_{\mathbb{L}_n}(\mu)$. Therefore we arrive at $\varphi_n(\mathscr{M}_{\mathbb{L}_n}(\lambda)) = 0$, i.e. there exists no nontrivial morphism $\varphi_n \colon \mathscr{M}_{\mathbb{L}_n}(\lambda) \longrightarrow \mathscr{M}_{\mathbb{L}_n}(\mu)$, whence $\mathbb{L}_n$ is semisimple. $\square$

We next consider the algebra $\mathbb{A}_n$. According to Proposition 2, if $\mathbb{A}_n$ is not semisimple then there exists the exact sequence

$$0 \longrightarrow \mathscr{N}_{\mathbb{A}_n}(\lambda) \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\lambda) \xrightarrow{\varphi_n} \mathscr{M}_{\mathbb{A}_n}(\mu), \tag{4.2}$$

where μ, λ are two partitions, such that $\lambda \vdash (n - t_\lambda)$ and $\mu \vdash (n - t_\mu)$, $t_\lambda < t_\mu$. In the next lemma we show that we can, without loss of generality, assume that $\lambda \vdash n$. Since $\mathscr{M}_{\mathbb{A}_n}(\lambda) \cong S^\lambda$ is irreducible this implies that we have an embedding $\varphi_n \colon S^\lambda \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\mu)$.

Lemma 2. *Suppose $x \neq 0$ and $\mathbb{A}_n$ is not semisimple. Then there exists $n_1 \leq n$, two partitions $\lambda_1 \vdash n_1$, $\mu_1 \vdash n_1 - t_1$ and the short exact sequence*

$$0 \longrightarrow S^{\lambda_1} \xrightarrow{\varphi_{n_1}} \mathscr{M}_{\mathbb{A}_{n_1}}(\mu_1). \tag{4.3}$$

Proof. If $\mathbb{A}_n$ is not semisimple, then there exists $\lambda \vdash (n - t_\lambda)$, $\mu \vdash (n - t_\mu)$, where $t_\lambda < t_\mu$ and the exact sequence of eq. (4.2). Without loss of generality we may assume $0 < t_\lambda$. Theorem 1 guarantees the existence of the embedding $e_\lambda \colon \mathscr{M}_{\mathbb{A}_{n-1}}(\lambda) \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\lambda)$ and $e_\mu \colon \mathscr{M}_{\mathbb{A}_{n-1}}(\mu) \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\mu)$ given by $e_\lambda(\mathfrak{a} \otimes v) = \mathfrak{a}\mathfrak{u}_n \otimes v$ and $e_\mu(\mathfrak{a} \otimes w) = \mathfrak{a}\mathfrak{u}_n \otimes w$, respectively. We shall show that $\varphi_n \colon \mathscr{M}_{\mathbb{A}_n}(\lambda) \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\mu)$ induces a nontrivial morphism of $\mathbb{A}_{n-1}$-left modules via

$$\begin{tikzcd} \mathscr{M}_{\mathbb{A}_n}(\lambda) \arrow[r, "\varphi_n"'] & \mathscr{M}_{\mathbb{A}_n}(\mu) \\ \mathscr{M}_{\mathbb{A}_{n-1}}(\lambda) \arrow[u, "e_\lambda"'] \arrow[r, dashed, "\varphi_{n-1}"] & \mathscr{M}_{\mathbb{A}_{n-1}}(\mu) \arrow[u, "e_\mu"'] \end{tikzcd} \tag{4.4}$$

Let $\mathfrak{a} \otimes v \in \mathscr{M}_{\mathbb{A}_{n-1}}(\lambda)$, where $\mathfrak{a} \in \mathscr{I}_{n-1}^{n-t_\lambda}$, $\mathrm{bot}(\mathfrak{a}) = \mathrm{bot}(\mathfrak{u}_{n-1,t_\lambda-1})$ and $v \in S^\lambda$. Since φ_n is a morphism of $\mathbb{A}_n$-left modules we have $\varphi_n(\mathfrak{u}_n \cdot e_\lambda(\mathfrak{a} \otimes v)) = \mathfrak{u}_n \cdot \varphi(e_\lambda(\mathfrak{a} \otimes v))$ and in view of $\varphi_n(\mathfrak{u}_n \cdot e_\lambda(\mathfrak{a} \otimes v)) = x\, \varphi_n(e_\lambda(\mathfrak{a} \otimes v))$ we derive

$$\mathfrak{u}_n \varphi_n(e_\lambda(\mathfrak{a} \otimes v)) = x\, \varphi_n(e_\lambda(\mathfrak{a} \otimes v)). \tag{4.5}$$

We represent $\varphi_n(e_\lambda(\mathfrak{a} \otimes v)) = \sum_r \mathfrak{a}_r \otimes v_r$, where the $\mathfrak{a}_r$ are distinct $\mathscr{I}_n^{n-t_\mu}$-diagrams, having noncrossing verticals, with $\mathrm{bot}(\mathfrak{a}_r) = \mathrm{bot}(\mathfrak{u}_{n,t_\mu})$ and $v_r \in S^\mu$. Then we obtain

$$x^{-1}\mathfrak{u}_n \varphi_n(e_\lambda(\mathfrak{a} \otimes v)) = x^{-1} \sum_r (\mathfrak{u}_n \mathfrak{a}_r) \otimes v_r = \sum_r \mathfrak{a}_r \otimes v_r. \tag{4.6}$$

Since different $\mathscr{I}_n^{n-t_\mu}$-diagrams are by construction linear independent we can conclude from eq. (4.6), that each $\mathfrak{a}_r$ has a loop at top-vertex n. Therefore there exists for each $\varphi_n(e_\lambda(\mathfrak{a}\otimes v)) = \sum_r \mathfrak{a}_r \otimes v_r$, a unique element $\sum_r \mathfrak{a}_r^\ddagger \otimes v_r \in \mathscr{M}_{\mathbb{A}_{n-1}}(\mu)$, obtained by removing the vertices n and n' and their corresponding loops from each $\mathfrak{a}_r$. Since $\mathscr{M}_{\mathbb{A}_{n-1}}(\lambda)$ is generated by tensors of the form $\mathfrak{a}\otimes v$, φ_n induces the mapping

$$\begin{aligned} \varphi_{n-1}\colon \mathscr{M}_{\mathbb{A}_{n-1}}(\lambda) &\longrightarrow \mathscr{M}_{\mathbb{A}_{n-1}}(\mu) \\ \mathfrak{a}\otimes v &\longmapsto \sum_r \mathfrak{a}_r^\ddagger \otimes v_j, \end{aligned} \tag{4.7}$$

with the property $e_\mu \cdot \varphi_{n-1} = \varphi_n \cdot e_\lambda$, i.e. φ_{n-1} makes the diagram in eq. (4.4) commutative. By construction, φ_{n-1} is a morphism of $\mathbb{A}_{n-1}$-left modules.

Claim. We have $w \in \mathscr{N}_{\mathbb{A}_{n-1}}(\lambda)$ if and only if $e_\lambda(w) \in \mathscr{N}_{\mathbb{A}_n}(\lambda)$.

Suppose first $w = \sum_i \mathfrak{a}_i \otimes v_i \notin \mathscr{N}_{\mathbb{A}_{n-1}}(\lambda)$. According to eq. (2.5), there exists some $\mathfrak{b}_0 \in \mathscr{I}_{n-1}^{n-t_\lambda}$ such that

$$\mathfrak{b}_0 \cdot \sum_i \mathfrak{a}_i \otimes v_i = \sum_i (\mathfrak{b}_0\mathfrak{a}_i)\otimes v_i = \widetilde{\mathfrak{b}}_0 \otimes \sum_i \delta_{\mathfrak{b}_0\mathfrak{a}_i} x^{\ell(\mathfrak{b}_0,\mathfrak{a}_i)} \sigma_{\mathfrak{b}_0,\mathfrak{a}_i} v_i \neq 0.$$

This equation implies in the $\mathbb{A}_n$-module $\mathscr{M}_{\mathbb{A}_n}(\lambda)$

$$\mathfrak{b}_0 \cdot \sum_i \mathfrak{a}_i \mathfrak{u}_n \otimes v_i = \sum_i (\mathfrak{b}_0 \mathfrak{u}_n \mathfrak{a}_i)\otimes v_i$$

where $\mathfrak{b}_0\mathfrak{u}_n \in \mathscr{I}_n^{n-t_\lambda}$. In view of $\ell(\mathfrak{b}_0\mathfrak{u}_n, \mathfrak{a}_i) = \ell(\mathfrak{b}_0,\mathfrak{a}_i)$ and $\mathfrak{b}_0\mathfrak{a}_i = x^{\ell(\mathfrak{b}_0,\mathfrak{a}_i)}\widetilde{\mathfrak{b}}_0\sigma_{\mathfrak{b}_0,\mathfrak{a}_i}$, where $\sigma_{\mathfrak{b}_0,\mathfrak{a}_i} \in S_{n-t_\lambda}$ we obtain

$$\mathfrak{b}_0 \cdot \sum_i \mathfrak{a}_i\mathfrak{u}_n \otimes v_i = \widetilde{\mathfrak{b}}_0\mathfrak{u}_n \otimes \sum_i \delta_{\mathfrak{b}_0\mathfrak{a}_i} x^{\ell(\mathfrak{b}_0,\mathfrak{a}_i)} \sigma_{\mathfrak{b}_0,\mathfrak{a}_i} v_i \neq 0.$$

I.e. we have shown $w \notin \mathscr{N}_{\mathbb{A}_{n-1}}(\lambda) \implies e_\lambda(w) \notin \mathscr{N}_{\mathbb{A}_n}(\lambda)$. Second suppose $e_\lambda(w) \notin \mathscr{N}_{\mathbb{A}_n}(\lambda)$. Then there exists some $\mathfrak{b}_0 \in \mathscr{I}_n^{n-t_\lambda}$ such that

$$\mathfrak{b}_0 \cdot \sum_i \mathfrak{a}_i\mathfrak{u}_n \otimes v_i = \sum_i (\mathfrak{b}_0\mathfrak{u}_n)\mathfrak{a}_i \otimes v_i \neq 0$$

and $\mathfrak{b}_0\mathfrak{u}_n$ is the scalar multiple of a diagram $\mathfrak{x} \in \mathscr{I}_n^{n-t_\lambda}$, having top$(\mathfrak{x})$ = top$(\mathfrak{b}_0)$ and a loop at n'. We accordingly compute

$$\mathfrak{b}_0 \cdot \sum_i \mathfrak{a}_i\mathfrak{u}_n \otimes v_i = x^s \sum_i \mathfrak{x}\mathfrak{a}_i \otimes v_i = x^s\, \tilde{\mathfrak{r}} \otimes \sum_i \delta_{\mathfrak{r}\mathfrak{a}_i} x^{\ell(\mathfrak{r},\mathfrak{a}_i)} \sigma_{\mathfrak{r},\mathfrak{a}_i} v_i \neq 0,$$

where $\tilde{\mathfrak{r}}$ is given by $\mathfrak{r}\mathfrak{a}_i = x^{\ell(\mathfrak{r},\mathfrak{a}_i)}\delta_{\mathfrak{r}\mathfrak{a}_i}\tilde{\mathfrak{r}}\,\sigma_{\mathfrak{r},\mathfrak{a}_i}$. We may assume that $\mathfrak{r}$ has a loop at top-vertex n, since this feature does not affect the term $w_1 = \sum_i \delta_{\mathfrak{r}\mathfrak{a}_i} x^{\ell(\mathfrak{r},\mathfrak{a}_i)} \sigma_{\mathfrak{r},\mathfrak{a}_i} v_i$. By construction, $\tilde{\mathfrak{r}}$ has then also a

loop at n and there exists a $\mathfrak{c} \in \mathscr{I}_{n-1}^{n-t_\lambda}$ with the property $\mathfrak{x} = \mathfrak{c}\mathfrak{u}_n$ in $\mathscr{I}_n^{n-t_\lambda}$. In view of $\delta_{\mathfrak{x}\mathfrak{a}_i} = \delta_{\mathfrak{c}\mathfrak{a}_i}$, $\sigma_{\mathfrak{x},\mathfrak{a}_i} = \sigma_{\mathfrak{c},\mathfrak{a}_i}$ and $\ell(\mathfrak{x},\mathfrak{a}_i) = \ell(\mathfrak{c},\mathfrak{a}_i)$ we obtain

$$\mathfrak{c} \cdot \sum_i \mathfrak{a}_i \otimes v_i = \sum_i (\mathfrak{c}\mathfrak{a}_i) \otimes v_i = \widetilde{\mathfrak{c}} \otimes \sum_i \delta_{\mathfrak{c}\mathfrak{a}_i} x^{\ell(\mathfrak{c},\mathfrak{a}_i)} \sigma_{\mathfrak{c},\mathfrak{a}_i} v_i = \widetilde{\mathfrak{c}} \otimes w_1 \neq 0.$$

That is, we have proved $e_\lambda(w) \notin \mathscr{N}_{\mathbb{A}_n}(\lambda) \implies w \notin \mathscr{N}_{\mathbb{A}_{n-1}}(\lambda)$ and the Claim follows.
Using $e_\mu \cdot \varphi_{n-1} = \varphi_n \cdot e_\lambda$, we can now immediately conclude $\ker(\varphi_{n-1}) = \mathscr{N}_{\mathbb{A}_{n-1}}(\lambda)$. Indeed, if $w \in \ker(\varphi_{n-1})$ then $e_\lambda(w) \in \ker(\varphi_n) = \mathscr{N}_{\mathbb{A}_n}(\lambda)$, whence $w \in \mathscr{N}_{\mathbb{A}_{n-1}}(\lambda)$. If $w \in \mathscr{N}_{\mathbb{A}_{n-1}}(\lambda)$, then $e_\lambda(w) \in \mathscr{N}_{\mathbb{A}_n}(\lambda)$, whence $e_\mu \cdot \varphi_{n-1}(w) = 0$, from which $\varphi_{n-1}(w) = 0$, i.e. $w \in \ker(\varphi_{n-1})$ follows. Therefore we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{N}_{\mathbb{A}_n}(\lambda) \arrow[r] & \mathscr{M}_{\mathbb{A}_n}(\lambda) \arrow[r, "\varphi_n"] & \mathscr{M}_{\mathbb{A}_n}(\mu) \\
0 \arrow[r] & \mathscr{N}_{\mathbb{A}_{n-1}}(\lambda) \arrow[r] \arrow[u, "e_\lambda"] & \mathscr{M}_{\mathbb{A}_{n-1}}(\lambda) \arrow[r, "\varphi_{n-1}"] \arrow[u, "e_\lambda"] & \mathscr{M}_{\mathbb{A}_{n-1}}(\mu). \arrow[u, "e_\mu"]
\end{tikzcd}$$

Iterating the above construction we arrive, after t_λ steps, at some $\lambda_1 \vdash n_1$, $\mu_1 \vdash (n-t_1)$ and the exact sequence $0 \longrightarrow \mathscr{N}_{\mathbb{A}_{n_1}}(\lambda_1) \longrightarrow \mathscr{M}_{\mathbb{A}_{n_1}}(\lambda_1) \longrightarrow \mathscr{M}_{\mathbb{A}_{n_1}}(\mu_1)$. Since $\lambda_1 \vdash n_1$ we have $\mathscr{N}_{\mathbb{A}_{n_1}}(\lambda_1) = 0$ and $\mathscr{M}_{\mathbb{A}_{n_1}}(\lambda_1) \cong S^{\lambda_1}$, whence Lemma 2. □

Our next result establishes further restrictions on the embedding $0 \longrightarrow S^\lambda \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\mu)$ in terms of the partition μ. Using the exact sequence for the restriction functor of Theorem 1 we shall prove the existence of such an embedding with the additional property $\mu_2 \vdash (n-1)$ or $\mu_2 \vdash (n-2)$. The result is in analogy to the Brauer algebra case proved in [4].

Lemma 3. *Suppose $\varphi_{n_1}\colon S^{\lambda_1} \longrightarrow \mathscr{M}_{\mathbb{A}_{n_1}}(\mu_1)$ is an embedding where $\lambda_1 \vdash n_1$ and $\mu_1 \vdash n_1 - t_1$. Then for $\mathbb{A}_n$ there exist $n_2 \leq n_1$, a pair of partitions (λ_2, μ_2) and an embedding $S^{\lambda_2} \longrightarrow \mathscr{M}_{\mathbb{A}_{n_2}}(\mu_2)$, such that $\lambda_2 \vdash n_2$, $\mu_2 \vdash (n_2-1)$ or $\mu_2 \vdash (n_2-2)$.*

Proof. Since $\mathrm{res}_{S_{n_1-1}}(S^{\lambda_1}) \cong \bigoplus_{\nu \sqsubset \lambda_1} S^\nu$ we obtain for some $\nu \sqsubset \lambda_1$, $\nu \vdash (n_1-1)$, the embedding $\varphi_\nu\colon S^\nu \longrightarrow \mathrm{res}_{n_1-1}(\mathscr{M}_{\mathbb{A}_{n_1}}(\mu_1))$. An interpretation of $\mathrm{res}_{n-1}(\mathscr{M}_{\mathbb{A}_{n_1}}(\mu_1))$ is given via Theorem 1 in terms of the exact sequence

$$0 \longrightarrow \bigoplus_{\alpha_1 \sqsubseteq \mu_1} \mathscr{M}_{n_1-1}(\alpha_1) \longrightarrow \mathrm{res}_{n_1-1}(\mathscr{M}_{\mathbb{A}_{n_1}}(\mu_1)) \longrightarrow \bigoplus_{\mu_1 \sqsubset \beta_1} \mathscr{M}_{n_1-1}(\beta_1) \longrightarrow 0. \tag{4.8}$$

Suppose we have $\varphi_\nu(S^\nu) \subset F_{n_1}(\mu_1) \cong \bigoplus_{\alpha_1 \sqsubseteq \mu_1} \mathscr{M}_{n_1-1}(\alpha_1)$. Then the irreducibility of S^ν implies the embedding $S^\nu \longrightarrow \mathscr{M}_{n_1-1}(\alpha_1)$ for some $\alpha_1 \sqsubseteq \mu_1$. Otherwise, we have $\varphi_\nu(S^\nu) \not\subset F_{n_1}(\mu_1)$. The

irreducibility of $\varphi_\nu(S^\nu)$ guarantees

$$(\varphi_\nu(S^\nu) \oplus F_{n_1}(\mu_1))\,/F_{n_1}(\mu_1) \cong \varphi_\nu(S^\nu).$$

In view of eq. (4.8) we have

$$\mathrm{res}_{n_1-1}(\mathscr{M}_{\mathbb{A}_{n_1}}(\mu_1))/F_{n_1}(\mu_1) \cong \bigoplus_{\mu_1 \sqsubset \beta_1} \mathscr{M}_{n_1-1}(\beta_1)$$

which implies an embedding $S^\nu \longrightarrow \mathscr{M}_{n_1-1}(\beta_1)$, for some $\mu_1 \sqsubset \beta_1$.
Therefore we have the following situation: each iteration of the above argument reduces the size of the partition $\lambda_1 \vdash n_1$ by one and an analogous reduction of the partition μ_1 can occur at most $(n_1 - t_1) < n_1$ times. Any further iteration cannot decrease the size of μ_1, while decreasing the size of λ_1. That is, iteration produces a pair (λ_2, μ_2) where $\lambda_2 \vdash n_2$ and $\mu_2 \vdash (n_2 - h)$, where $h = 1$ or $h = 2$. Indeed, for $h = 2$, i.e. $\mu_2 \vdash n_2 - 2$, further reduction can generate the trivial embedding $S^\nu \longrightarrow S^\nu$, i.e. we derive, using the above notation, $\nu = \beta_1$, for $\nu \sqsubset \lambda_2$. Therefore further reduction is in general not possible and we have shown that iteration of the above process leads to a pair of partitions (λ_2, μ_2) with the properties $\lambda_2 \vdash n_2$ and $\mu_2 \vdash (n_2 - 1)$ or $\mu_2 \vdash (n_2 - 2)$. $\square$

Now we are in position to prove our main result:

Theorem 4. *Suppose $x \neq 0$. If $x \notin \mathbb{Z}$, then the algebra $\mathbb{A}_n$ is semisimple.*

Proof. According to Proposition 2, if $\mathbb{A}_n$ is not semisimple there exists a nontrivial morphism $\varphi_n\colon \mathscr{M}_{\mathbb{A}_n}(\lambda) \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\mu)$ with $\ker(\varphi_n) = \mathscr{N}_{\mathbb{A}_n}(\lambda)$.
In view of Lemma 2 and Lemma 3 we can, without loss of generality, assume that there exists an embedding $\varphi_n\colon S^\lambda \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\mu)$, where $\lambda \vdash n$ and either $\mu \vdash (n-1)$ or $\mu \vdash (n-2)$. According to Proposition 1, $\mathscr{N}_{\mathbb{A}_n}(\mu)$ is the unique maximal $\mathscr{M}_{\mathbb{A}_n}(\mu)$-submodule. Therefore $\varphi_n(S^\lambda) \subset \mathscr{N}_{\mathbb{A}_n}(\mu)$, i.e. we have the embedding $\varphi_n\colon S^\lambda \longrightarrow \mathscr{N}_{\mathbb{A}_n}(\mu)$. In the following we distinguish the two cases $\mu \vdash (n-1)$ and $\mu \vdash (n-2)$.
Case 1: $\mu \vdash (n-1)$. We prove that $x \neq 0$ implies $\mathscr{N}_{\mathbb{A}_n}(\mu) = 0$. Let $\mathfrak{a} \in \mathscr{I}_n^{n-1}$, where $\mathrm{bot}(\mathfrak{a}) = \mathrm{bot}(\mathfrak{u}_{n,1})$ and let $v \in S^\mu$. For any $\mathfrak{a} \otimes v \in \mathscr{M}_{\mathbb{A}_n}(\mu)$, there exists some $\sigma_0 \in S_{n-1}$ and some index $1 \leq j \leq n$ such that $\mathfrak{a}_j = \mathfrak{a}\sigma_0$ has noncrossing vertical arcs and has its unique, top-vertex loop at j. $\mathfrak{a}_j$ has the property $\mathfrak{a} \otimes v = \mathfrak{a}_j \otimes \sigma_0^{-1} v$ and any $u \in \mathbb{I}_n^{n-1}\mathfrak{u}_{n,1} \otimes_{S_{n-1}} S^\mu$ can be written as $u = \sum_j \mathfrak{a}_j \otimes w_j$. Let $U_1 = \sum_i \mathfrak{u}_i$. Then $U_1 \in \mathbb{I}_n^{n-1}$ and any $\mathfrak{a}_j$ satisfies the eigenvector equation $U_1 \cdot \mathfrak{a}_j = x\, \mathfrak{a}_j$. Let $u = \sum_j \mathfrak{a}_j \otimes w_j \in \mathscr{N}_{\mathbb{A}_n}(\mu)$. Since $U_1 \in \mathbb{I}_n^{n-1}$, the action of U_1 on $\mathscr{N}_{\mathbb{A}_n}(\mu)$ is

trivial, i.e.

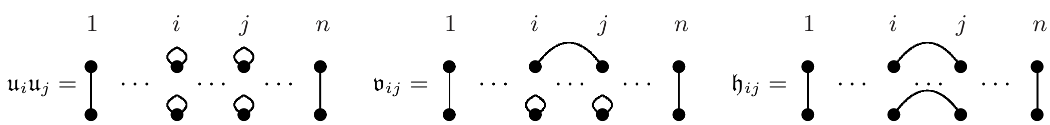
$$U_1 \cdot u = \sum_j (U_1 \cdot \mathfrak{a}_j) \otimes w_j = \sum_i x\, \mathfrak{a}_j \otimes w_j = x\, u = 0, \tag{4.9}$$

which implies, in view of $x \neq 0$, $\mathscr{N}_{\mathbb{A}_n}(\mu) = 0$.

Case 2: $\mu \vdash (n-2)$. For each diagram $\mathfrak{a} \in \mathscr{I}_n^{n-2}$, such that $\mathrm{bot}(\mathfrak{a}) = \mathrm{bot}(\mathfrak{u}_{n,2})$ there exist a pair of indices, $i < j$ and a permutation $\sigma_0 \in S_{n-2}$ such that either $\mathfrak{a}\sigma_0 = \mathfrak{a}_{i,j}^{\cap}$ or $\mathfrak{a}\sigma_0 = \mathfrak{a}_{i,j}^{\circ}$ holds. Here $\mathfrak{a}_{i,j}^{\cap} \in \mathscr{I}_n^{n-2}$ has noncrossing verticals, a horizontal arc connecting i and j and $\mathrm{bot}(\mathfrak{a}_{i,j}^{\cap}) = \mathrm{bot}(\mathfrak{u}_{n,2})$. Analogously, $\mathfrak{a}_{i,j}^{\circ} \in \mathscr{I}_n^{n-2}$ has noncrossing verticals, two loops at i, j and $\mathrm{bot}(\mathfrak{u}_{n,2})$. We can write each tensor $\mathfrak{a} \otimes w$, where $\mathfrak{a} \in \mathscr{I}_n^{n-2}$ with $\mathrm{bot}(\mathfrak{a}) = \mathrm{bot}(\mathfrak{u}_{n,2})$ and $w \in S^{\mu}$, uniquely as either $\mathfrak{a}_{i,j}^{\cap} \otimes \sigma_0^{-1} w$ or $\mathfrak{a}_{i,j}^{\circ} \otimes \sigma_0^{-1} w$. Let $g \colon \mathscr{M}_{\mathbb{A}_n}(\mu) \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\mu)$ be the involution given via linear extension of $g(\mathfrak{a}_{i,j}^{\circ} \otimes w) = \mathfrak{a}_{i,j}^{\cap} \otimes w$ and $g(\mathfrak{a}_{i,j}^{\cap} \otimes w) = \mathfrak{a}_{i,j}^{\circ} \otimes w$. Furthermore, let $\mathfrak{v}_{i,j} \in \mathscr{I}_n^{n-2}$ be the diagram having straight verticals except of a horizontal arc connecting the top-vertices i, j and two loops at the bottom vertices i', j', respectively. We introduce

$$U_2 = \sum_{i<j} \mathfrak{u}_i \mathfrak{u}_j,\ V_2 = \sum_{i<j} \mathfrak{v}_{i,j} \quad \text{and} \quad H_2 = \sum_{i<j} \mathfrak{h}_{i,j}, \tag{4.10}$$

where $\mathfrak{h}_{i,j} \in \mathscr{I}_n^{n-2}$ has straight vertical arcs except of the top-vertices i, j and bottom-vertices i', j', which are connected by a horizontal arc, respectively. We observe $U_2, V_2, H_2 \in \mathbb{I}_n^{n-2}$.



As for the action of U_2, a routine computation yields $U_2 \cdot \mathfrak{a}_{i,j}^{\cap} = x\, \mathfrak{a}_{i,j}^{\circ}$ and $U_2 \cdot \mathfrak{a}_{i,j}^{\circ} = x^2\, \mathfrak{a}_{i,j}^{\circ}$. Similarly we obtain for V_2, $V_2 \cdot \mathfrak{a}_{i,j}^{\cap} = x\, \mathfrak{a}_{i,j}^{\cap}$ and $V_2 \cdot \mathfrak{a}_{i,j}^{\circ} = x^2\, \mathfrak{a}_{i,j}^{\cap}$. Let $\tau_{(i,j)}$ act on the diagram $\mathfrak{a}_{i,j}^{\cap}$ as the transposition $(i,j) \in S_n$ from the left and $\tilde{\tau}_{(a,b)}$ as transposition $(a,b) \in S_{n-2}$, from the right, respectively. Then

$$H_2 \cdot \mathfrak{a}_{i,j}^{\circ} \quad = \quad x\, \mathfrak{a}_{i,j}^{\cap} \tag{4.11}$$

$$H_2 \cdot \mathfrak{a}_{i,j}^{\cap} \quad = \quad \left((x-1) + \sum_{i<j} \tau_{(i,j)} - \sum_{a<b} \tilde{\tau}_{(a,b)} \right) \mathfrak{a}_{i,j}^{\cap}, \tag{4.12}$$

where eq. (4.12) holds according to [4], Lemma 2, p.655. We write an element $v \in \mathscr{N}_{\mathbb{A}_n}(\mu)$ as

$$v = \sum_{i,j} \mathfrak{a}_{i,j}^{\cap} \otimes r_{i,j} + \sum_{i,j} \mathfrak{a}_{i,j}^{\circ} \otimes s_{i,j}$$

and set $v^\cap = \sum_{i,j} \mathfrak{a}_{i,j}^\cap \otimes r_{i,j}$ and $v^\circ = \sum_{i,j} \mathfrak{a}_{i,j}^\circ \otimes s_{i,j}$. Since $(H_2 - x^{-1}V_2) \in \mathbb{I}_n^{n-2}$, we obtain

$$\begin{aligned}(H_2 - x^{-1}V_2) \cdot (v^\cap + v^\circ) &= H_2 \cdot v^\cap + H_2 \cdot v^\circ - x^{-1}V_2 \cdot v^\cap - x^{-1}V_2 \cdot v^\circ \\ &= H_2 \cdot v^\cap + xg(v^\circ) - v^\cap - x\,g(v^\circ) \\ &= H_2 \cdot v^\cap - v^\cap.\end{aligned}$$

Suppose now there exists some $0 \neq v_0 \in \varphi_n(S^\lambda) \subset \mathscr{N}_{\mathbb{A}_n}(\mu)$ such that $v_0^\cap \neq 0$ and $v_0^\circ \neq 0$. Since $\varphi_n(S^\lambda)$ is an irreducible S_n-module and the S_n-action cannot change a horizontal arc into a pair of loops, for any $0 \neq v \in \varphi_n(S^\lambda) \subset \mathscr{N}_{\mathbb{A}_n}(\mu)$, $v^\cap \neq 0$ and $v^\circ \neq 0$ holds. Therefore if there exits some $0 \neq v_0 \in \varphi_n(S^\lambda) \subset \mathscr{N}_{\mathbb{A}_n}(\mu)$ such that $v_0^\cap \neq 0$ and $v_0^\circ \neq 0$, then we have for any $0 \neq v \in \varphi_n(S^\lambda)$, $(H_2 - 1) \cdot v^\cap = 0$, i.e.

$$\left((x-1) + \sum_{i<j} \tau_{(i,j)} - \sum_{a<b} \tilde{\tau}_{(a,b)} - 1\right) \cdot v^\cap = 0. \tag{4.13}$$

We proceed by studying the action of $\sum_{i<j} \tau_{(i,j)}$ and $\sum_{a<b} \tilde{\tau}_{(a,b)}$ on the set

$$\varphi_n^\cap(S^\lambda) = \{v^\cap \mid v^\cap + v^\circ \in \varphi(S^\lambda)\}. \tag{4.14}$$

The $\mathbb{A}_n$-module $\mathscr{M}_{\mathbb{A}_n}(\mu)$ can be regarded as a $S_n \times S_{n-2}$-left module via

$$(\sigma, \sigma') \cdot (\mathfrak{a} \otimes w) = \sigma \cdot (\mathfrak{a} \otimes \sigma' w) \tag{4.15}$$

and $\sigma \cdot (\mathfrak{a} \otimes \sigma' w) = (\sigma \mathfrak{a} \sigma') \otimes w$ shows that the action of eq. (4.12) and eq. (4.15) coincide. Furthermore, $\varphi_n(S^\lambda)$ becomes via eq. (4.15) a $S_n \times S_{n-2}$-submodule of $\mathscr{M}_{\mathbb{A}_n}(\mu)$ and induces an $S_n \times S_{n-2}$ action on the set $\varphi_n^\cap(S^\lambda)$ via $(\sigma, \sigma') \cdot (\mathfrak{a}_{i,j}^\cap \otimes w) = \sigma \cdot (\mathfrak{a}_{i,j}^\cap \otimes \sigma' w)$. Accordingly, $\varphi_n^\cap(S^\lambda)$ can be considered as a $S_n \times S_{n-2}$-module and the projection

$$\pi_1 \colon \varphi_n(S^\lambda) \longrightarrow \varphi_n^\cap(S^\lambda), \quad (v^\cap + v^\circ) \mapsto v^\cap, \tag{4.16}$$

establishes an isomorphism of $S_n \times S_{n-2}$-modules. Indeed, only injectivity needs to be proved. Using $x \neq 0$, $U_2 \in \mathbb{I}_n^{n-2}$ and $(v^\cap + v^\circ) \in \mathscr{N}_{\mathbb{A}_n}(\lambda)$, injectivity follows from

$$x^{-1}U_2 \cdot (v^\cap + v^\circ) = g(v^\cap) + xv^\circ = 0,$$

i.e. $v^\circ = -x^{-1}g(v^\cap)$. Obviously, $\sum_{i<j} \tau_{(i,j)}$ and $\sum_{a<b} \tilde{\tau}_{(a,b)}$ are contained in the centers of the group algebras $F[S_n]$ and $F[S_{n-2}]$, respectively and Schur's Lemma implies that they act as homotheties on irreducible representations. Since $\varphi_n(S^\lambda)$ embeds into the $S^\lambda \otimes S^\mu$-component of $\mathscr{M}_{\mathbb{A}_n}(\mu)$, the particular values of $\sum_{i<j} \tau_{(i,j)}$ and $\sum_{a<b} \tilde{\tau}_{(a,b)}$ are given by [12]

$$\sum_{i<j} \tau_{(i,j)} = \sum_{p \in [\lambda]} c(p) \quad \text{and} \quad \sum_{a<b} \tilde{\tau}_{(a,b)} = \sum_{p \in [\mu]} c(p). \tag{4.17}$$

Since $\varphi_n(S^\lambda) \subset \mathscr{N}_{\mathbb{A}_n}(\mu)$ we obtain

$$\forall\, v^\cap \in \varphi_n^\cap(S^\lambda); \qquad \left((x-1) + \sum_{p\in[\lambda]} c(p) - \sum_{p\in[\mu]} c(p) - 1\right) v^\cap = 0, \tag{4.18}$$

which implies

$$(x-1) + \sum_{p\in[\lambda]} c(p) - \sum_{p\in[\mu]} c(p) - 1 = 0. \tag{4.19}$$

Since the content $c(p)$ is an integer, eq. (4.19) implies $x \in \mathbb{Z}$. It thus remains to consider the cases $v^\cap = 0$ or $v^\circ = 0$. The case of $v^\circ = 0$ is due to [4]. In analogy we derive, using the action of H_2 on $\varphi_n(S^\lambda)$

$$\forall\, v \in \varphi_n(S^\lambda); \qquad H_2 \cdot v = \left((x-1) + \sum_{p\in[\lambda]} c(p) - \sum_{p\in[\mu]} c(p)\right) \cdot v = 0, \tag{4.20}$$

which implies $(x-1)+\sum_{p\in[\lambda]} c(p)-\sum_{p\in[\mu]} c(p) = 0$. This immediatly allows us to conclude $x \in \mathbb{Z}$. In case of $v^\cap = 0$ we obtain for any $v \in \varphi_n(S^\lambda)$

$$U_2 \cdot v = x^2 v = 0, \tag{4.21}$$

which is, in view of $x \neq 0$ impossible.

We have therefore showed that in case of $\mu \vdash (n-1)$, $x \neq 0$ implies $\mathscr{N}_{\mathbb{A}_n}(\mu) = 0$. Since $\mathscr{N}_{\mathbb{A}_n}(\mu)$ is the unique, maximal $\mathscr{M}_{\mathbb{A}_n}(\lambda)$-submodule, there cannot exist an embedding $\varphi_n \colon S^\lambda \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\mu)$. In case of $\mu \vdash (n-2)$, our proof guarantees that for $x \notin \mathbb{Z}$, there exists no embedding $\varphi_n \colon S^\lambda \longrightarrow \mathscr{M}_{\mathbb{A}_n}(\mu)$, whence $\mathbb{A}_n$ is semisimple. □

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